General Tensors

Tensors which are expressed in terms of components referred to 3-dimensional space are called cartesian tensors.

Tensors which are expressed in terms of components referred to n-dimensional co-ordinate systems are called general tensors.

Here we shall discuss the general general tensors.

n-Dimensional Space

A point in n-dimensional space is denoted by $(x^1, x^2, x^3 \ldots x^n)$, where 1, 2, 3 ... n are superscripts but not exponents

Note # In general tensors it is more convenient to write the co-ordinates $(x_1, x_2 \ldots x_n)$ as $(x^1, x^2, x^3 \ldots x^n)$

Transformation of Co-ordinates

Let $(x^1, x^2, x^3 \ldots x^n)$ and $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n)$ be the co-ordinate of point in X and $\bar{X}$ systems respectively suppose that there exist n independent relations between the co-ordinates of the two systems in the form

$$\bar{x}^1 = \bar{x}^1(x^1, x^2 \ldots x^n)$$

$$\bar{x}^2 = \bar{x}^2(x^1, x^2, x^3 \ldots x^n)$$

$$\bar{x}^3 = \bar{x}^3(x^1, x^2, x^3 \ldots x^n)$$

$$\bar{x}^n = \bar{x}^n(x^1, x^2, x^3 \ldots x^n)$$

which can be written as transformation

$$T: \quad \bar{x}^j = \bar{x}^j(x^1, x^2 \ldots x^n) \longrightarrow (1)$$

where it is supposed that the functions involved are single-valued, continuous and have continuous derivatives. Als functions are revesible.

The relation (1) defines a transformation of co-ordinates from X-system to $\bar{X}$-system

Since the transformation T in (1) is assumed to be reversible and one-one, we can write

$$\bar{T}^1 \quad x^j = x^j(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n) \longrightarrow (2)$$

which defines a transformation of co-ordinates from $\bar{X}$-system to X-system. Here $x^j(\bar{x})$ are single valued.

To ensure the satisfaction of conditions we have imposed on the transformation of co-ordinates, it will be sufficient that the

Functions in ① i.e $\bar{x}^j(x_i)$ are continuous together with their 1st partial derivatives in some region R of n-dimensional space and that the Jacobian determinant $J \equiv \left|\frac{\partial \bar{x}^j}{\partial x^j}\right|$ does not vanish at any point of the region R. It would follow under these conditions that not only single valued inverse ② exists, but the functions $x^j(\bar{x})$ in ② are also of class C^1 in some neighbourhood of the point under consideration.

We observe that if the functions $\bar{x}^j(x)$ in ① are of class C^1, then by Taylor's formula, we have

$$\bar{x}^j = a^j_0 + a^j_i x^i$$

where a^j_i is the value of $\frac{\partial \bar{x}^j}{\partial x^i}$ evaluated at some point P' of region R

Thus the transformation in ① with conditions stated above, is locally linear. The non-vanishing of Jacobian guarantees that this system of linear equations has a unique solution

Note # Symbol C^n is used to denote the class of functions which are continuous together with

their 1st order partial derivatives

Example # As an example of transformation of the type stated consider a system of equations giving relation between the spherical coordinates x^i and Cartesian co-ordinates $\bar{x}^i$

$$T: \begin{cases} \bar{x}^1 = x^1 \sin x^2 \cos x^3 \\ \bar{x}^2 = x^1 \sin x^2 \sin x^3 \\ \bar{x}^3 = x^1 \cos x^2 \end{cases}$$

If we suppose that $x^1 > 0$, $0 < x^2 < \pi$, $0 \leq x^3 < 2\pi$, then $J \neq 0$ and the inverse transformation is given by

$$\bar{T}: \begin{cases} x^1 = +\sqrt{(\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2} \\ x^2 = \operatorname{Tan}^{-1}\left(\dfrac{\sqrt{(\bar{x}^1)^2 + (\bar{x}^2)^2}}{\bar{x}^3} \right) \\ x^3 = \operatorname{Tan}^{-1}\left(\dfrac{\bar{x}^2}{\bar{x}^1} \right) \end{cases}$$

By Muhammad Hussain Lecturer (Maths) Govt College Asghar Mall Rawalpindi.

Tensor form of Transformation Equations Between Cartesian Co-ordinates and Polar Cylinderical and Spherical Co-ordinates

(a) Cartesian & Polar Co-ordinates #

The transformation equations between the Cartesian Co-ordinates (x, y) and the polar Co-ordinates (r, θ) are

$x = r\cos\theta$ and $r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\theta = \tan^{-1}(y/x)$

Let $(x, y) = (x^1, x^2)$ $(r, \theta) = (\bar{x}^1, \bar{x}^2)$

Then the above transformations equations are

$x^1 = \bar{x}^1\cos\bar{x}^2$ and $\bar{x}^1 = \sqrt{(x^1)^2+(x^2)^2}$

$x^2 = \bar{x}^1\sin\bar{x}^2$ $\bar{x}^2 = \tan^{-1}\left(\frac{x^2}{x^1}\right)$

(b) Cartesian and Cylinderical Polar Co-ordinates

The transformation equations between the cartesian and cylinderical polar Co-ordinates (r, θ, z) are $r = \sqrt{x^2+y^2+z^2}$

$x = r\cos\theta$ $\theta = \tan^{-1}(y/x)$

$y = r\sin\theta$

$z = z$ $z = z$

Let $(x, y, z) = (x^1, x^2, x^3)$ $(r, \theta, z) = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$

Then the above transformation equations are

$x^1 = \bar{x}^1\cos\bar{x}^2$ $\bar{x}^1 = \sqrt{(x^1)^2+(x^2)^2}$

$x^2 = \bar{x}^1 \sin \bar{x}^2$ $\qquad \bar{x}^2 = \tan^{-1}\left(\frac{x^2}{x^1}\right)$

$x^3 = \bar{x}^3$ $\qquad \bar{x}^3 = x^3$

(C) Cartesian & Spherical Co-ordinates

The transformation equations between the Cartesian (x, y, z) and the spherical polar Co-ordinates (r, θ, ϕ) are

$x = r \sin\theta \cos\phi$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$y = r \sin\theta \sin\phi$ $\qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$z = r \cos\theta$ $\qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

Let $(x, y, z) = (x^1, x^2, x^3)$ and $(r, \theta, \phi) = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$

Then the above transformations equations are

$x^1 = \bar{x}^1 \sin\bar{x}^2 \cos\bar{x}^3$ $\qquad \bar{x}^1 = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$

$x^2 = \bar{x}^1 \sin\bar{x}^2 \sin\bar{x}^3$ $\qquad \bar{x}^2 = \tan^{-1}\left(\frac{\sqrt{(x^1)^2+(x^2)^2}}{x^3}\right)$

$x^3 = \bar{x}^1 \cos\bar{x}^2$ $\qquad \bar{x}^3 = \tan^{-1}\left(\frac{x^2}{x^1}\right)$

By M. Husain Lecturer (Maths) Govt. College Asghar Mall.

Tensor Notation For Matrices

(a) Lower-Index Matrix Notation

We know that a matrix of order $m \times n$ in ordinary subscript notation is denoted as

$$[a_{ij}]_{mn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is customary to denote the determinant of $[a_{ij}]_{mn}$ as $|a_{ij}|$

(b) Mixed-Index Matrix Notation

A matrix of order $m \times n$ in the mixed index notation (one upper index showing row and one lower index showing column) is written as

$$[a^i_j]_{mn} = \begin{bmatrix} a^1_1 & a^1_2 & a^1_3 & \cdots & a^1_n \\ a^2_1 & a^2_2 & a^2_3 & \cdots & a^2_n \\ \vdots & & & & \vdots \\ a^m_1 & a^m_2 & \cdots & & a^m_n \end{bmatrix}$$

det of $[a^i_j]_{m \times n}$ is denoted by $|a^i_j|$

(C) Upper-Index Matrix Notation

A matrix of order $m \times n$ in the upper-index notation is denoted as

$$[a^{ij}]_{mn} = \begin{bmatrix} a^{11} & a^{12} & a^{13} & \cdots & a^{1n} \\ a^{21} & a^{22} & a^{23} & \cdots & a^{2n} \\ \vdots & & & & \vdots \\ a^{m1} & a^{m2} & \cdots & & a^{mn} \end{bmatrix}$$

By Muhammad Hussain Lecturer (Maths)
Govt College Asghar Mall.

The Kronecker Delta δ_j^i

The Kronecker delta or substitution operator, denoted by δ_j^i is defined as

$$\delta_j^i = \begin{cases} 1 & \text{if } i=j \\ 0 & i \neq j \end{cases}$$

$\Rightarrow \delta_1^1 = \delta_2^2 = \delta_3^3 = \cdots = \delta_n^n = 1$

and $\delta_2^1 = 0 \quad \delta_3^2 = 0 \quad \delta_5^3 = 0$

Clearly $\delta_j^i = \delta_i^j$ for all i and j

It is also denoted by δ_{ij}

Einstein Summation Convention#

Consider an expression

$$a_1 x^1 + a_2 x^2 + \cdots + a_n x^n = \sum_{j=1}^{n} a_j x^j$$

According Einstein Convention summation sign is omitted and it may be written as

$$a_j x^j$$

Instead of j we can use any other letter.

Einstein summation Convention is as follows

" Whenever an index appears twice (twice as a subscript, or twice as superscript or once as a subscript and once as super) in an expression, then summation w.r.t that index is understood.

for all values of index; unless otherwise stated.

Dummy Index

An index which repeated in a given expression so that Summation Convention applies is called a dummy index e.g in expression $a_j x^j$, the index j is a dummy index

Free Index

An index occurring only once in an expression is called a free index e.g in expression

$$a_{ij} x^j = b_i \quad i = 1, 2 \ldots n$$

then index j is a dummy index and i is a free index which represents n equations.

Double Sums

An expression can involve more than one index. e.g

$$a_{ij} x^j x^i$$

indicates Summation on i and j simultaneously. If an expression has two summation indices, then there will be n^2 terms in the sum and if there are three indics, then there will be n^3 terms in the summation.

Question# Write down the expression $a_{ij} x^i x^j$ in full.

Sol# The given expression will have n^2 terms in the sum

holding j fixed and summing over i, we have n terms in sum as

$$a_{ij} x^i x^j = a_{1j} x^1 x^j + a_{2j} x^2 x^j + \cdots + a_{nj} x^n x^j$$

Now each term on R.H.S has a dummy index j. Hence

$$a_{ij} x^i x^j = (a_{11} x^1 x^1 + a_{12} x^1 x^2 + a_{13} x^1 x^3 + \cdots + a_{1n} x^n x^1) + (a_{21} x^2 x^1 + a_{22} x^2 x^2 + \cdots + a_{2n} x^2 x^n) + \cdots + (a_{n1} x^n x^1 + a_{n2} x^n x^2 + \cdots + a_{nn} x^n x^n)$$

Question# Write the following using summation Convention#

$$g_{11} dx^1 dx^1 + g_{21} dx^2 dx^1 + g_{31} dx^3 dx^1 + g_{12} dx^1 dx^2 + g_{22} dx^2 dx^2 + g_{32} dx^3 dx^1 + g_{13} dx^1 dx^3 + g_{23} dx^2 dx^3 + g_{33} dx^3 dx^3$$

Sol# The given expression can be written as

$$(g_{11} dx^1 dx^1 + g_{21} dx^2 dx^1 + g_{31} dx^3 dx^1) + (g_{12} dx^1 dx^2 + g_{22} dx^2 dx^2 + g_{32} dx^3 dx^2) + (g_{13} dx^1 dx^3 + g_{23} dx^2 dx^3 + g_{33} dx^3 dx^3)$$

$$= g_{p1} dx^p dx^1 + g_{p2} dx^p dx^2 + g_{p3} dx^p dx^3$$

$$= g_{pq} dx^p dx^q \qquad p = 1,2,3 \quad q = 1,2,3$$

Substitutions

Suppose we wants to substitute

$$\bar{x}^i = a_{ij} x^i$$ in equation

$$Q = b_{ij} \bar{x}^i x^j$$

Then, simple substitution will given an absurd expression

$$Q = b_{ij} a_{ij} x^j x^j$$

The correct procedure for substitution is as. First identify any dummy indices in the expression to be substituted that coincide with indices occurring in the main (indices) expression. Change these dummy indices to indices not occurring in the main expression, we may carry substitution in usual manner as follows.

Step-I # $\bar{x}^i = a_{ij} x^j$ $\quad Q = b_{ij} \bar{x}^i x^j$

We see that dummy index j is duplicated

Step-II # Changing dummy index j to k we get

$$\bar{x}^i = a_{ik} x^k$$

Step-III Substitute and rearrange to get :

$$Q = b_{ij}(a_{ir}x^r)x^j = a_{ir}b_{ij}x^r x^j$$

By M. Hussain Lecturer (Maths) Govt. College Asghar Mall

Question# if $\bar{x}^i = a_{ij}x^j$, express the Quadratic form $Q = g_{ij}\bar{x}^i\bar{x}^j$ in terms of x-variables.

Sol# dummy j is duplicated. First we write

$$\bar{x}^i = a_{ir}x^r \qquad \bar{x}^j = a_{js}x^s$$

Then by substution we have

$$Q = g_{ij}(a_{ir}x^r)(a_{js}x^s)$$
$$= g_{ij}a_{ir}a_{js}x^r x^s$$

Question# Using Kronecker delta, Calculate

$$\delta^i_j x^i x^j$$

Solution# First holding j fixed and summing over i we have

$$\delta^i_j x^i x^j = \delta^1_j x^1 x^j + \delta^2_j x^2 x^j + \cdots \delta^n_j x^n x^j$$

Now each term on R.H.S has dummy index j

$$\delta^i_j x^i x^j = \delta^1_1 x^1 x^1 + \delta^1_2 x^1 x^2 + \delta^1_3 x^1 x^3 + \cdots \delta^1_n x^1 x^n$$
$$+ \delta^2_1 x^2 x^1 + \delta^2_2 x^2 x^2 + \delta^2_3 x^2 x^3 + \cdots \delta^2_n x^2 x^n$$
$$+ \cdots\cdots\cdots\cdots\cdots$$
$$+ \delta^n_1 x^n x^1 + \delta^n_2 x^n x^2 + \cdots + \delta^n_n x^n x^n$$
$$= x^1x^1 + x^2x^2 + x^3x^3 + \cdots + x^nx^n = x^i x^i$$
$$i = 1, 2, 3 \cdots n$$

Theorem# Show that if $x^1, x^2, x^3, \ldots x^n$ are independent variables, Then

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j$$

Proof# If $i = j$, Then

$$\frac{\partial x^i}{\partial x^j} = 1 \quad \text{Since } x^i = x^j$$

If $i \neq j$ then

$$\frac{\partial x^i}{\partial x^j} = 0 \quad \text{Since } x^i, x^j \text{ are}$$

independent variables

Thus

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases}$$

By M. Husain Lecturer (Maths) Asghar Mall College

Theorem# Prove that

(i)# $$\frac{\partial x^i}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^k} = \delta^i_k$$

(ii)# $$\frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^k} = \delta^i_k$$

Proof# (i) The co-ordinates x^i are functions of $\bar{x}^j$ which are in turn function of x^k.

By chain rule we have

$$x^i = x^i[\bar{x}^j(x^k)]$$

$$\frac{\partial x^i}{\partial x^k} = \frac{\partial x^i}{\partial \bar{x}^j} \frac{\partial \bar{x}^j}{\partial x^k}$$

But $\frac{\partial x^i}{\partial x^k} = \delta^i_k$

$\Rightarrow \frac{\partial x^i}{\partial \bar{x}^j} \cdot \frac{\partial \bar{x}^j}{\partial x^k} = \delta^i_k$

(ii) Considering functional relation.

$$\bar{x}^i = \bar{x}^i [x^j (\bar{x}^k)]$$

and applying chain rule we have

$$\frac{\partial \bar{x}^i}{\partial \bar{x}^k} = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^k}$$

$$\delta^i_k = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^k}$$

Theorem# Prove that

$$\delta^i_j A^j = A^i$$

OR Prove that δ^i_j is a substitution operator

Proof# Index j is dummy index and we have

$$\delta^i_j A^j = \delta^i_1 A^1 + \delta^i_2 A^2 + \delta^i_3 A^3 + \dots + \delta^i_n A^n$$

when $i = 1$

$$\delta^1_j A^j = \delta^1_1 A^1 + 0 + 0 + 0 + \dots + 0$$

$$= A^1$$

when $i = 2$

$$\delta^2_j A^j = A^2$$

when $i = n$

$$\delta^n_j A^j = \delta^n_1 A^1 + \delta^n_2 A^2 + \dots + \delta^n_n A^n$$
$$= 0 + 0 + \dots + A^n$$
$$= A^n$$

Thus in all cases we have

$$\delta^i_j A^j = A^i$$

We note that δ^i_j operating on A^j has substituted the free index i for dummy j and this justifies that δ^i_j is an substitution operator.

By M. Hussain Lecturer Govt College Asahar Mall

Theorem# Prove that $\delta^i_k \delta^k_j = \delta^i_j$

Proof# We note that on L.H.S index k represents the summation and we have

$$\delta^i_k \delta^k_j = \delta^i_1 \delta^1_j + \delta^i_2 \delta^2_j + \delta^i_3 \delta^3_j + \dots \delta^i_n \delta^n_j$$

when $j = 1$

$$\delta^i_k \delta^k_j = \delta^i_1 (1) + \delta^i_2 (0) + \delta^i_3 (0) + \dots \delta^i_n (0)$$
$$= \delta^i_1$$

when $j = 2$

$$\delta^i_k \delta^k_2 = \delta^i_1 \delta^1_2 + \delta^i_2 (1) + 0 + 0 + \dots 0$$
$$= \delta^i_2$$

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when $j = n$ $\quad \delta^i_k \delta^k_n = \delta^i_n$

It follows therefore that

$$\delta^i_k \delta^k_j = \delta^i_j$$

Similarly we can prove that

$$\delta^i_j \delta^k_i = \delta^k_j = \delta^j_k$$

Question# For n arbitrary, Show that

(i) $\delta^i_i = n$ (ii) $\delta^i_j \delta^i_j = n$

(iii) $\delta^i_j \delta^j_h \delta^k_i = n$ (iv) $\delta^i_j \delta^k_l A_{ik} = A_{jl}$

(v) $\delta^i_j \delta^j_h A_{ih} = A_{ii}$

Sol# (i) we have

$$\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \cdots + \delta^n_n$$

$$= 1 + 1 + 1 + \cdots + 1 \quad (n \text{ terms})$$

$$= n$$

(ii) $\delta^i_j \delta^i_j = \delta^1_j \delta^1_j + \delta^2_j \delta^2_j + \cdots \delta^n_j \delta^n_j$

$$= (\delta^1_1 \delta^1_1 + \delta^1_2 \delta^1_2 + \cdots + \delta^1_n \delta^1_n)$$

$$+ (\delta^2_1 \delta^2_1 + \delta^2_2 \delta^2_2 + \delta^2_2 \delta^2_3 + \cdots + \delta^2_n \delta^2_n)$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ (\delta^n_1 \delta^n_1 + \delta^n_2 \delta^n_2 + \delta^n_3 \delta^n_3 + \cdots + \delta^n_n \delta^n_n)$$

$$= (1)(1) + (1)(1) + \cdots + (1)(1)$$

$$= n$$

(iii) $\delta^i_j \delta^j_k \delta^k_i = \delta^i_k \delta^k_i = \delta^i_i = n$

(iv) $\delta^i_j \delta^k_l A_{ik} = \delta^i_j A_{il}$

$= A_{jl}$

(v) $\delta^i_j \delta^j_k A_{ik} = \delta^i_k A_{ik} = A_{ii}$

$= A_{11} + A_{22} + \dots + A_{nn}$

By M.Hussain Lecturer Govt College Asghar Mall.

The Jacobian

Consider the co-ordinate transformation from X-system to $\bar{X}$-system defined by

$$\bar{x}^j = \bar{x}^j(x^1, x^2, x^3, \dots x^n) \quad j = 1, 2 \dots n$$

Then the n^2 1st partial derivatives $\longrightarrow$ ①

$\frac{\partial \bar{x}^j}{\partial x^i}$ arising from ① are arranged in an $n \times n$ matrix J

$$J = \left[\frac{\partial \bar{x}^j}{\partial x^i}\right] = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \frac{\partial \bar{x}^1}{\partial x^3} & \dots & \frac{\partial \bar{x}^1}{\partial x^n} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \frac{\partial \bar{x}^2}{\partial x^3} & \dots & \frac{\partial \bar{x}^2}{\partial x^n} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{\partial x^2} & \dots & \dots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix}$$

The matrix J is called Jacobian matrix

and its determinant, $|\bar{J}|$ is called Jacobian of the transformation ①

If the transformation ① has the inverse transformation given by

$$x^j = x^j(\bar{x}^1, \bar{x}^2, \bar{x}^3 \dots\dots \bar{x}^n) \longrightarrow ②$$

$$j = 1, 2, \dots n$$

Then the jacobian matrix $\bar{J}$ of ② is defined by

$$\bar{J} = \left[\frac{\partial x^j}{\partial \bar{x}^i}\right] = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \dots & \frac{\partial x^1}{\partial \bar{x}^n} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} & \dots & \frac{\partial x^2}{\partial \bar{x}^n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial x^n}{\partial \bar{x}^1} & \frac{\partial x^n}{\partial \bar{x}^2} & \dots & \frac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix}$$

The determinant of $\bar{J}$, $|\bar{J}|$ is called the jacobian of the transformation ②

By Muhammad Hussain Lecturer
Govt. College Asghar Mall
R.W.P

Theorem # If a transformation of Co-ordinates $T: \bar{x}^j = \bar{x}^j(x^1, x^2 \ldots x^n)$ Possesses an inverse Transformation T^{-1} $x^j = x^j(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$ and $|J|$, $|\bar{J}|$ be Jacobians of the transformation, then prove that

$$|J|\,|\bar{J}| = 1$$

Proof # Since

$$\bar{x}^j = \bar{x}^j(x^1, x^2, x^3, \ldots\ldots x^n) \longrightarrow (1)$$

and.

$$x^j = x^j(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots\ldots, \bar{x}^n) \longrightarrow (2)$$

Therefore

$$|J| = \left|\frac{\partial \bar{x}^j}{\partial x^i}\right| \text{ and } |\bar{J}| = \left|\frac{\partial x^j}{\partial \bar{x}^i}\right|$$

pulting (2) in (1) we get

$$\bar{x}^j = \bar{x}^j\left[x^1(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n), x^2(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n), \ldots x^n(\bar{x}^1 \ldots \bar{x}^n)\right]$$

Differentiating w.r.t $\bar{x}^i$ we get

$$\frac{\partial \bar{x}^j}{\partial \bar{x}^i} = \frac{\partial \bar{x}^j}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^i} \qquad k = 1, 2, \ldots n$$

$$\delta^j_i = \frac{\partial \bar{x}^j}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^i}.$$

Taking the determinant on both sides

$$|\delta^j_i| = \left|\frac{\partial \bar{x}^j}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^i}\right| \overset{20}{=} \left|\frac{\partial \bar{x}^j}{\partial x^k}\right| \left|\frac{\partial x^k}{\partial \bar{x}^i}\right|$$

$$1 = |J|\,|\bar{J}| \quad \text{proved.}$$

Theorem# If transformations

$T_1 \quad \bar{x}^i = \bar{x}^i(x^1, x^2, x^3, \ldots x^n)$

and

$T_2 \quad z^i = z^i(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots \bar{x}^n)$

are real single, valued reversible, Then prove that the Jacobian of the product transformation $T_3 = T_2 T_1$ given by

$T_3 \quad z^i = z^i[\bar{x}^1(x^1 x^2 \ldots x^n), \ldots \bar{x}^n(x^1 x^2 \ldots x^n)]$

is equal to the product of the transformations.

Proof# Let $|J_1|, |J_2|, |J_3|$ be jacobians of transformations T_1, T_2, T_3 respectively.

$$z^i = z^i[\bar{x}^1(x^1 x^2 \ldots x^n), \bar{x}^2(x^1 x^2 \ldots x^n) \ldots \bar{x}^n(x^1 x^2 \ldots x^n)]$$

$$J_3 = \frac{\partial z^i}{\partial x^k} = \frac{\partial z^i}{\partial \bar{x}^j} \cdot \frac{\partial \bar{x}^j}{\partial x^k}$$

$$|J_3| = \left|\frac{\partial z^i}{\partial \bar{x}^j} \cdot \frac{\partial \bar{x}^j}{\partial x^k}\right|$$

$$= \left|\frac{\partial z^i}{\partial \bar{x}^j}\right| \cdot \left|\frac{\partial \bar{x}^j}{\partial x^k}\right|$$

$$= |J_2|\,|J_1| \quad \text{proved}$$

Question# Suppose the co-ordinate transformation from the system $(x^1, x^2, \ldots x^n)$ to the system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$ is

$$\bar{x}^1 = x^1 x^2 \qquad \bar{x}^2 = (x^2)^2$$

(a) Calculate the Jacobian matrix J and $|J|$ of the transformation

(b) Obtain the equations for inverse transformation

(c) Calculate the jacobian matrix $\bar{J}$ and compare it with J^{-1}

Solution# (a) Given that

$$\bar{x}^1 = x^1 x^2 \qquad \bar{x}^2 = (x^2)^2 \longrightarrow (1)$$

$$\frac{\partial \bar{x}^1}{\partial x^1} = x^2 \qquad \frac{\partial \bar{x}^1}{\partial x^2} = x^1$$

$$\frac{\partial \bar{x}^2}{\partial x^1} = 0 \qquad \frac{\partial \bar{x}^2}{\partial x^2} = 2x^2$$

The Jacobian matrix J is given by

$$J = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} \end{bmatrix} = \begin{bmatrix} x^2 & x^1 \\ 0 & 2x^2 \end{bmatrix}$$

$$= 2(x^2)^2$$

and the Jacobian of the transformation is

$$|J| = \begin{vmatrix} x^2 & x^1 \\ 0 & 2x^2 \end{vmatrix} = 2(x^2)^2$$

(b) Solving the transformation equations

(i) for x^1 and x^2 we get

$$x^1 = \frac{\bar{x}^1}{\sqrt{\bar{x}^2}} \qquad x^2 = \sqrt{\bar{x}^2} \longrightarrow (2)$$

(c) we have $\frac{\partial x^1}{\partial \bar{x}^1} = \frac{1}{\sqrt{\bar{x}^2}} \quad \frac{\partial x^1}{\partial \bar{x}^2} = \frac{-\bar{x}^1}{2(\bar{x}^2)^{3/2}}$

$$\frac{\partial x^2}{\partial \bar{x}^1} = 0 \qquad \frac{\partial x^2}{\partial \bar{x}^2} = \frac{1}{2\sqrt{\bar{x}^2}}$$

Therefore the Jacobian matrix $\bar{J}$ of the inverse transformation is given by

$$\bar{J} = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\bar{x}^2}} & \frac{-\bar{x}^1}{2(\bar{x}^2)^{3/2}} \\ 0 & \frac{1}{2\sqrt{\bar{x}^2}} \end{bmatrix}$$

$$\bar{J}^{-1} = \frac{1}{|J|} \text{Adj } J = \frac{1}{2(x^2)^2} \begin{bmatrix} 2x^2 & -x^1 \\ 0 & x^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x^2} & -\frac{x^1}{2(x^2)^2} \\ 0 & \frac{1}{2x^2} \end{bmatrix}$$

using (2)

$$\bar{J}^1 = \begin{bmatrix} \frac{1}{\sqrt{\bar{x}^2}} & -\frac{x^1}{2(x^2)^2} \\ 0 & \frac{1}{2\sqrt{\bar{x}^2}} \end{bmatrix} = \bar{J}$$

Question# Consider a system of linear equations of the form $\bar{x}^i = A_{ij} x^j$ and suppose that $[B_{ij}]$ is a matrix of numbers such that for all i and j

$$B^{ik} A_{kj} = \delta^i_j$$ (i.e the matrix B^{ij} is inverse of matrix A_{ij}). Solve the system for x^i in terms of $\bar{x}^i$

Solution# We have $\bar{x}^i = A_{ij} x^j$

Multiplying both sides by B^{ki} and summing over i

$$B^{ki} \bar{x}^i = B^{ki} A_{ij} x^j$$
$$= \delta^k_j x^j = x^k$$
$$\text{or } x^i = B^{ij} \bar{x}^j$$

Question # if $a^i_j A^j = c A^i$ is a tensor equation, where c is a scalar and A^j an arbitrary vector. Show that $a^i_j = \delta^i_j c$

Solution # Given that

$$a^i_j A^j = c A^i$$
$$= c \delta^i_j A^j$$
$$\Rightarrow (a^i_j - c \delta^i_j) A^j = 0$$

Since A^j is an arbitrary vector, therefore

$$a^i_j - c \delta^i_j = 0 \text{ or } a^i_j = c \delta^i_j$$

By Muhammad Hussain Lecturer Govt College Asghar Ma

Transformation by Covariance and Contravariance

Consider a continuously differentiable function $f(x^1, x^2, \ldots x^n)$ which represents the scalar $f(P)$ and a transformation of co-ordinates

$$T: \quad x^i = x^i(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n) \rightarrow (1)$$

If we form a set of n partial derivatives

$$\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots \frac{\partial f}{\partial x^n} \text{ or } \{f_{x^i}\} \longrightarrow (2)$$

Now question arises that what happens with the set $\{f_{x^i}\}$ when the co-ordinates x^i transform according to (1). These fractions do not automatically "become" anything unitill we state the law which will be used in calculating the corresponding functions in $\bar{x}$-frame. e-g we can put in each function $f_{x^i}(x^1, x^2 \ldots x^n)$, the values of x_is from (1) ; this will give a set of n functions

$$g_1(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n), g_2(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n) \ldots g_n(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n) \rightarrow (3)$$

But if we have the components of a gradient of $f(P)$, then the set of functions corresponding to (2) is not (3) but the set of n partial derivatives.

$\frac{\partial f}{\partial \bar{x}^1}, \frac{\partial f}{\partial \bar{x}^2}, \ldots \frac{\partial f}{\partial \bar{x}^n}$

Computed by the rule for differentiation of composite function i.e

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^i} \quad (i, \alpha = 1, 2, \ldots n)$$

This is called covariant law

Now consider a set of n differentials

$$dx^1, dx^2, dx^3, \ldots dx^n \longrightarrow (4)$$

where $x^1, x^2, x^3 \ldots x^n$ are related to variables $\bar{x}^i$ by relation (1). If we have two points $P_1(x^1, x^2 \ldots x^n)$ and $P_2(x^1+dx^1, x^2+dx^2, \ldots x^n+dx^n)$, then set of n numbers in (4) determines the displacement vector from P_1 to P_2

The same displacement vector when referred to $\bar{X}$-system has components

$$d\bar{x}^1, d\bar{x}^2, d\bar{x}^3, \ldots d\bar{x}^n$$

where

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^\alpha} dx^\alpha \quad (i, \alpha = 1, 2, \ldots n)$$

This rule is called Contravariant law.

By Muhammad Hussain Lecturer
Govt College Asghar Mall RWP.

Contravariant and Covariant Vector.

A set of n quantities A^i in a co-ordinate system $(x^1, x^2, \ldots x^n)$ form the components of a contravariant tensor of 1st order if it is related to another set of n quantities $\bar{A}^p$ in a co-ordinates system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$ by the transformation law

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \longrightarrow (1)$$

Equation (1) can be written in n equations

$$\bar{A}^1 = \frac{\partial \bar{x}^1}{\partial x^1} A^1 + \frac{\partial \bar{x}^1}{\partial x^2} A^2 + \ldots \frac{\partial \bar{x}^1}{\partial x^n} A^n$$

$$\bar{A}^2 = \frac{\partial \bar{x}^2}{\partial x^1} A^1 + \frac{\partial \bar{x}^2}{\partial x^2} A^2 + \ldots \frac{\partial \bar{x}^2}{\partial x^n} A^n$$

$$\bar{A}^n = \frac{\partial \bar{x}^n}{\partial x^1} A^1 + \frac{\partial \bar{x}^n}{\partial x^2} A^2 + \ldots + \frac{\partial \bar{x}^n}{\partial x^n} A^n$$

In matrix form

$$\begin{bmatrix} \bar{A}^1 \\ \bar{A}^2 \\ \vdots \\ \bar{A}^n \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \ldots & \frac{\partial \bar{x}^1}{\partial x^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{\partial x^2} & \ldots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix} \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^n \end{bmatrix}$$

A set of n quantities A_i in a co-ordinate system $(x^1, x^2 \ldots x^n)$ is said to form the components of a covariant tensor of 1st order if it is related to another set of n quantities $\bar{A}_p$ in a co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$ by transformation law

$$\bar{A}_p = \frac{\partial x^i}{\partial \bar{x}^p} A_i \longrightarrow (2)$$

note that superscript is used to denote contravariant components and subscript is used to denote covariant components

Equation (2) gives n equations as

$$\bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 + \ldots \frac{\partial x^n}{\partial \bar{x}^1} A_n$$

$$\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} A_1 + \frac{\partial x^2}{\partial \bar{x}^2} A_2 + \ldots \frac{\partial x^n}{\partial \bar{x}^2} A_n$$

$$\bar{A}_n = \frac{\partial x^1}{\partial \bar{x}^n} A_1 + \frac{\partial x^2}{\partial \bar{x}^n} A_2 + \ldots \frac{\partial x^n}{\partial \bar{x}^n} A_n$$

which may be written in matrix form as.

$$\begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \vdots \\ \bar{A}_n \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^1} & \cdots & \frac{\partial x^n}{\partial \bar{x}^1} \\ \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots & \frac{\partial x^n}{\partial \bar{x}^2} \\ \vdots & & & \\ \frac{\partial x^1}{\partial \bar{x}^n} & \frac{\partial x^2}{\partial \bar{x}^n} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}$$

Covariant Tensor of order Two

A set of n^2 quantities A_{ij} in a co-ordinate system $(x^1, x^2, x^3 \cdots x^n)$ is said to form the components of a covariant tensor of order two if it is related to another set of n^2 quantities $\bar{A}_{pq}$ in a co-ordinate system $(\bar{x}^1, \bar{x}^2 \cdots \bar{x}^n)$ by the transformation law

$$\bar{A}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$$

In matrix form

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \cdots \bar{A}_{1n} \\ \bar{A}_{21} & \bar{A}_{22} \cdots \bar{A}_{2n} \\ \vdots & \\ \bar{A}_{n1} & \bar{A}_{n2} \cdots A_{nn} \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^1} & \cdots & \frac{\partial x^n}{\partial \bar{x}^1} \\ \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots & \frac{\partial x^n}{\partial \bar{x}^2} \\ \vdots & & & \\ \frac{\partial x^1}{\partial \bar{x}^n} & \frac{\partial x^2}{\partial \bar{x}^n} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix} \times$$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \cdots & \frac{\partial x^1}{\partial \bar{x}^n} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^2} & \cdots & \frac{\partial x^2}{\partial \bar{x}^n} \\ \vdots & & & \\ \frac{\partial x^n}{\partial \bar{x}^1} & \frac{\partial x^n}{\partial \bar{x}^2} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix}$$

Contravariant Tensor of order Two

A set of n^2 quantities A^{ij} in a co-ordinate system $(x^1, x^2 \ldots x^n)$ is said to form the components of a contravariant tensor of 2nd order (or rank two) if it is related to another set of n^2 quantities $\bar{A}^{pq}$ in co-ordinate system $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$ by transformation law

$$\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}$$

In matrix form equations can be written as

$$\begin{bmatrix} \bar{A}^{11} & \bar{A}^{12} & \cdots & \bar{A}^{1n} \\ \bar{A}^{21} & \bar{A}^{22} & \cdots & \bar{A}^{2n} \\ \vdots & & & \\ \bar{A}^{n1} & \bar{A}^{n2} & \cdots & \bar{A}^{nn} \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \cdots & \frac{\partial \bar{x}^1}{\partial x^n} \\ \vdots & & & \\ \frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{\partial x^2} & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix} \times$$

$$\begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1n} \\ A^{21} & A^{22} & \cdots & A^{2n} \\ \vdots & & & \\ A^{n1} & A^{n2} & \cdots & A^{nn} \end{bmatrix} \times \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^1} & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \\ \vdots & & & \\ \frac{\partial \bar{x}^1}{\partial x^n} & \frac{\partial \bar{x}^2}{\partial x^n} & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix}$$

Mixed Tensor of order Two

A set of n^2 quantities A^i_j in a Co-ordinate System $(x^1, x^2 \ldots x^n)$ is said to form components of a mixed tensor of order two if it is related to another set of n^2 quantites $\bar{A}^p_q$ in a Co-ordinate system $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$ by transformation law

$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} A^i_j$$

In matrix form

$$\begin{bmatrix} \bar{A}^1_1 & \bar{A}^1_2 & \cdots & \bar{A}^1_n \\ \cdots & \cdots & \cdots & \cdots \\ \bar{A}^n_1 & \bar{A}^n_2 & \cdots & \bar{A}^n_n \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \cdots & \frac{\partial \bar{x}^1}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{\partial x^2} & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix} \times$$

$$\begin{bmatrix} A^1_1 & A^1_2 & \cdots & A^1_n \\ A^2_1 & A^2_2 & \cdots & A^2_n \\ \cdots & \cdots & \cdots & \cdots \\ A^n_1 & A^n_2 & \cdots & A^n_n \end{bmatrix} \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \cdots & \frac{\partial x^1}{\partial \bar{x}^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x^n}{\partial \bar{x}^1} & \frac{\partial x^n}{\partial \bar{x}^2} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix}$$

By Muhammad Hussain Lecturer Govt Asghar Mall College RWP.

Order of a Tensor#

Order indicates the no. of components in a given dimension It is indicated by the total number of superscripts or/and subscripts

Mixed Tensor

A set of n^{r+s} quantities $A^{j_1 j_2 \ldots j_s}_{i_1 i_2 \ldots i_r}(x)$ in X-co-ordinate system is a mixed tensor Co-variant of rank r and Contravariant of rank s provided that Corresponding quantities $\bar{A}^{p_1 p_2 \ldots p_s}_{q_1 q_2 \ldots q_r}(x)$ in $\bar{X}$-system are given by

$$\bar{A}^{p_1 p_2 \ldots p_s}_{q_1 q_2 \ldots q_r} = \frac{\partial x^{i_1}}{\partial \bar{x}^{q_1}} \frac{\partial x^{i_2}}{\partial \bar{x}^{q_2}} \ldots \frac{\partial x^{i_r}}{\partial \bar{x}^{q_r}} \cdot \frac{\partial \bar{x}^{p_1}}{\partial x^{j_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{j_2}} \ldots \frac{\partial \bar{x}^{p_s}}{\partial x^{j_s}} \times A^{j_1 j_2 \ldots j_s}_{i_1 i_2 \ldots i_r}$$

A Property of Law of Transformation of Tensors

Let the components of a mixed tensor in the X-system be denoted by $A^{j_1 j_2 \ldots j_s}_{i_1 i_2 \ldots i_r}(x)$ and its components

in $\bar{x}$-System by $\bar{A}^{j_1 j_2 \cdots j_s}_{i_1 i_2 \cdots i_r}(\bar{x})$

Then by Law of transformation we have

$$\bar{A}^{p_1 p_2 \cdots p_s}_{q_1 q_2 \cdots q_r}(\bar{x}) = \frac{\partial \bar{x}^{i_1}}{\partial x^{q_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{q_2}} \cdots \frac{\partial \bar{x}^{i_r}}{\partial x^{q_r}} \times \frac{\partial x^{p_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{p_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{p_s}}{\partial \bar{x}^{j_s}} A^{j_1 j_2 \cdots j_s}_{i_1 i_2 \cdots i_r} \longrightarrow (1)$$

on the other hand if we are given $\bar{A}^{p_1 p_2 \cdots p_s}_{q_1 q_2 \cdots q_r}$ in $\bar{x}$-system, then Components of the same tensor $A^{j_1 j_2 \cdots j_s}_{i_1 i_2 \cdots i_r}$ can be determined by formula.

$$A^{j_1 j_2 \cdots j_s}_{i_1 i_2 \cdots i_r} = \frac{\partial \bar{x}^{q_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{q_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{q_r}}{\partial x^{i_r}} \times \frac{\partial x^{j_1}}{\partial \bar{x}^{p_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{p_2}} \cdots \frac{\partial x^{j_s}}{\partial x^{p_s}} \bar{A}^{p_1 \cdots p_s}_{q_1 \cdots q_r} \longrightarrow (2)$$

Question # Prove That

(i) If A^i and B^j are two Contravariant vectors. Then $A^i B^j$ is a Contravariant tensors of order 2.

(ii) If A_i and B_j are two Covariant vectors, then $A_i B_j$ is a Covariant tensor of order 2

(iii) If A^i & B_j are Contravariant and Covariant vectors, then $A^i B_j$ is a mixed tensor of order 2

Sol # (i) Let $C^{ij} = A^i B^j$ and $\bar{A}^p$ and $\bar{B}^q$ be corresponding components of the given contravariant vectors in system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$ Then their laws of transformation are

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \longrightarrow (1)$$

$$\bar{B}^q = \frac{\partial \bar{x}^q}{\partial x^j} B^j \longrightarrow (2)$$

Multiplying (1) & (2)

$$\bar{A}^p \bar{B}^q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \cdot A^i B^j$$

$$\bar{C}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \cdot C^{ij} \longrightarrow (3)$$

Equation (3) shows that $C^{ij} = A^i B^j$ obeys the law of transformation of a contravariant tensor of order 2. Hence $A^i B^j$ is a contravariant tensor of order 2

(ii) Let $C_{ij} = A_i B_j$. Let $\bar{A}_p$ & $\bar{B}_q$ be corresponding components of the given covariant vectors in the system $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$, then their laws of transformation are

$$\bar{A}_p = \frac{\partial x^i}{\partial \bar{x}^p} A^i \longrightarrow (1)$$

$$\bar{B}_q = \frac{\partial x^j}{\partial \bar{x}^q} B^j \longrightarrow (2)$$

Multiplying ① & ②

$$\bar{A}_p \bar{B}_q = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A_i B_j$$

$$\bar{C}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} C_{ij} \longrightarrow ③$$

Where $\bar{C}_{pq} = \bar{A}_p \bar{B}_q$

Equation ③ shows that $C_{ij} = A_i B_i$ is a Covariant tensor of order 2

(iii) Let $C^i_j = A^i B_j$ and let $\bar{A}^p$ and $\bar{B}_q$ be the corresponding components of the given contravariant and covariant vectors respectively in the system $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$, then their Laws of transformation are

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \longrightarrow ①$$

$$\bar{B}_q = \frac{\partial x^j}{\partial \bar{x}^q} B_j \longrightarrow ②$$

Multiplying ① & ②

$$\bar{A}^p \bar{B}_q = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i B_j$$

$$\bar{C}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} C^i_j \longrightarrow ③$$

③ Shows that $C^i_j = A^i B_j$ is a mixed tensor of order 2

Theorem# Prove that the kronecker delta δ^i_j is a mixed tensor of the 2nd order.

Proof# If δ^i_j is a mixed tensor of rank 2, then it must transform according to the law

$$\bar{\delta}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} \delta^i_j$$

Now $$\bar{\delta}^p_q = \frac{\partial \bar{x}^p}{\partial \bar{x}^q} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^q}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} \cdot \frac{\partial x^i}{\partial x^j}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} \delta^i_j$$

$\Rightarrow \delta^i_j$ is a mixed tensor of rank 2

Similarly δ_{ij} tranform as the components of co-variant tensors and δ^{ij} transform as the components of a contravariant tensor.

Zeroth-Order Tensors (or Scalars)

Suppose ϕ is a function of co-ordinates $(x^1, x^2, \ldots, x^n)$ and let $\bar{\phi}$ denote the functional value under transformation to new set of co-ordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots \bar{x}^n)$. Then ϕ is called a scalar or invariant w.r.t co-ordinate transformation if it doesnot

when the co-ordinate system is changed i.e if

$$\Phi = \bar{\Phi}$$

Thus a tensor of rank zero is called a scalar or invariant.

M. Hussain Lecturer (Maths) Asghar Mall College.

Contravariant Tensor of Rank r

A set of n^r quantities $A^{i_1 i_2 i_3 \ldots i_r}$ in a co-ordinate system $(x^1, x^2, x^3, \ldots x^n)$ is said to form the components of a contravariant tensor of rank r if it is related to another set of n^r quantities $\bar{A}^{p_1 p_2 \ldots p_r}$ in co-ordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots \bar{x}^n)$ by the transformation equations

$$\bar{A}^{p_1 p_2 \ldots p_r} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \ldots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} A^{i_1 i_2 \ldots i_r}$$

Covariant Tensor of Rank r

A set of n^r quantities $A_{i_1 i_2 \ldots i_r}$ in a co-ordinate system $(x^1, x^2, \ldots x^n)$ is said to form the components of a covariant tensor of rank r if it is related to another set of n^r quantities $\bar{A}_{p_1 p_2 \ldots p_r}$ in a co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$ by the transformation equation

$$\bar{A}_{p_1 p_2 \cdots p_r} = \frac{\partial x^{i_1}}{\partial \bar{x}^{p_1}} \frac{\partial x^{i_2}}{\partial \bar{x}^{p_2}} \cdots \frac{\partial x^{i_r}}{\partial \bar{x}^{p_r}} A_{i_1 i_2 \cdots i_r}$$

Mixed Tensor of Rank $r+s$

A set of n^{r+s} quantities

$$A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$$ in a Co-ordinate system $(x^1, x^2 \cdots x^n)$ is said to form the components of a ~~covariant~~ mixed tensor of rank $r+s$, Contravarient of rank r, and Covariant of rank s if it is related to another set of n^{r+s} quantities $\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s}$ in a Co-ordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \cdots \bar{x}^n)$ by the transformation equation

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \times A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$$

Tensor Fields

If to each point of a region R in n-dimensional space there correspoinds a definite tensor, we say that a tensor field has been defined. This is a vector field or a scalar field according as the

the tensor is of rank one or zero

Note# To distinguish tensors defined over a region of space from tensors whose domain of definition is a single point, one occasionally speaks of a former as constituting a tensor field.

By M. Hussain Lecturer (Math) Govt College Asghar Mall

Question: Write transformation laws for the tensors (a) A^i_{jk} (b) B^{ijk}_{lm} (c) C^{ij}_{klmn}

Sol#

(a) Let $\bar{A}^{p}_{qr}$ be the components of the given tensor in the co-ordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \cdots \bar{x}^n)$

Then

$$\bar{A}^{p}_{qr} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\cdot\frac{\partial x^k}{\partial \bar{x}^r} A^i_{jk}$$

(b) Let $\bar{B}^{pqr}_{st}$ be components in $(\bar{x}^1, \bar{x}^2 \cdots \bar{x}^n)$ system, then

$$\bar{B}^{pqr}_{st} = \frac{\partial \bar{x}^p}{\partial x^i}\cdot\frac{\partial x^q}{\partial x^j}\cdot\frac{\partial x^r}{\partial x^k}\cdot\frac{\partial x^l}{\partial \bar{x}^s}\cdot\frac{\partial x^m}{\partial \bar{x}^t} B^{ijk}_{lm}$$

(c) Let $\bar{C}^{pq}_{rstu}$ be components in $\bar{x}$-system, then

$$\bar{C}^{pq}_{rstu} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j}\cdot\frac{\partial x^k}{\partial \bar{x}^r}\cdot\frac{\partial x^l}{\partial \bar{x}^s}\cdot\frac{\partial x^m}{\partial \bar{x}^t}\cdot\frac{\partial x^n}{\partial \bar{x}^u} C^{ij}_{klmn}$$

Inverse Transformation Laws For the First and 2nd order Tensors

Theorem (i) If $\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$, then prove that

$$A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i$$

(ii) # if $\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j$, then prove that

$$A_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{A}_i$$

Proof # (i) Given that

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$$

Multiplying both sides of the equation by $\frac{\partial x^k}{\partial \bar{x}^i}$

$$\frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i = \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial \bar{x}^i}{\partial x^j} A^j$$

$$= \frac{\partial x^k}{\partial x^j} A^j$$

$$= \delta^k_j A^j = A^k$$

$$A^k = \frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i$$

Replacing k by j, we get

$$A^k = \frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i$$

$$\Rightarrow A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i \quad \text{(As required)}$$

(ii) Given that $\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j$

Multiplying by $\frac{\partial \bar{x}^i}{\partial x^k}$

$$\frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^j}{\partial \bar{x}^i} A_j$$

$$= \frac{\partial x^j}{\partial x^k} A_j$$

$$= \delta^j_k A_j = A_k$$

Replacing k by j

$$A_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{A}_i$$

Theorem# (i) If $\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}$, then prove that

$$A^{ij} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} \bar{A}^{pq}$$

(ii) If $\bar{A}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$, then prove that

$$A_{ij} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} A_{pq}$$

(iii) If $\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i_j$, then prove that

$$A^i_j = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \bar{A}^p_q$$

Proof# (i) Given that $\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}$

Multiplying by $\frac{\partial x^k}{\partial \bar{x}^p} \cdot \frac{\partial x^m}{\partial \bar{x}^q}$

$$A = \frac{\partial x^k}{\partial \bar{x}^p} \cdot \frac{\partial x^m}{\partial \bar{x}^q} \cdot \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}$$

$$= \delta_i^k \delta_j^m A^{ij} = \delta_i^k A^{im} = A^{km}$$

$$A^{km}$$

$$\frac{\partial x^k}{\partial \bar{x}^p} \cdot \frac{\partial x^m}{\partial \bar{x}^q} \bar{A}^{pq} = \frac{\partial x^k}{\partial \bar{x}^p} \cdot \frac{\partial x^m}{\partial \bar{x}^q} \cdot \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}$$

$$= \delta_i^k \delta_j^m A^{ij}$$

$$= \delta_i^k A^{im} = A^{km}$$

$$\Rightarrow A^{km} = \frac{\partial x^k}{\partial \bar{x}^p} \cdot \frac{\partial x^m}{\partial \bar{x}^q} \bar{A}^{pq}$$

Replacing k by i & m by j

$$A^{ij} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} \bar{A}^{pq}$$

(ii) we have

$$A^{ij} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} \bar{A}^{pq}$$

(ii) Given that $\bar{A}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$

Multiplying by $\frac{\partial \bar{x}^p}{\partial x^k} \cdot \frac{\partial \bar{x}^q}{\partial x^m}$ we get

$$\frac{\partial \bar{x}^p}{\partial x^k} \cdot \frac{\partial \bar{x}^q}{\partial x^m} \bar{A}_{pq} = \frac{\partial \bar{x}^p}{\partial x^k} \cdot \frac{\partial \bar{x}^q}{\partial x^m} \cdot \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$$

$$= \delta_k^i \delta_m^j A_{ij}$$

$$= A_{km}$$

$$\Rightarrow A_{lm} = \frac{\partial \bar{x}^p}{\partial x^l} \cdot \frac{\partial \bar{x}^q}{\partial x^m} \bar{A}_{pq}$$

Replacing l by i & m by j we get

$$A_{ij} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \bar{A}_{pq}.$$

(iii) Given that

$$\bar{A}^{p}_{q} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i_j$$

Multiplying by $\frac{\partial x^l}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^q}{\partial x^m}$

$$\frac{\partial x^l}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^q}{\partial x^m} \bar{A}^{p}_{q} \Rightarrow \frac{\partial x^l}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^q}{\partial x^m} \cdot \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i_j.$$

$$= \delta^l_i \cdot \delta^j_m A^i_j.$$

$$= \delta^l_i A^i_m = A^l_m$$

$$A^l_m = \frac{\partial x^l}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^q}{\partial x^m} \cdot \bar{A}^{p}_{q}$$

Replacing l by i and m by j

$$A^i_j = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \bar{A}^{p}_{q} \quad \text{(as Required)}$$

Partial Derivative of 1st order Tensors#

Theorem# If A^i is a contravariant vector, show that the partial derivatives $\frac{\partial A^i}{\partial x^j}$ transform according to the rule

$$\frac{\partial \bar{A}^p}{\partial \bar{x}^q} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} + \frac{\partial^2 \bar{x}^p}{\partial x^i \partial x^j} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i$$

Proof# Let the Co-ordinate transformation be

$$\bar{x}^j = \bar{x}^j(x^1, x^2, x^3 - - - - x^n)$$

$$j = 1, 2 - - - n$$

Let $\bar{A}^p$ be the corresponding components of the Contravariant vector A^i in the Co-ordinate system $(\bar{x}^1, \bar{x}^2 - - -, \bar{x}^n)$, then the Law of transformation is

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \longrightarrow (1)$$

Differentiating w.r.t $\bar{x}^q$

$$\frac{\partial \bar{A}^p}{\partial \bar{x}^q} = \frac{\partial}{\partial \bar{x}^q}\left(\frac{\partial \bar{x}^p}{\partial x^i} A^i\right)$$

$$= \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial A^i}{\partial \bar{x}^q} + A^i \frac{\partial}{\partial \bar{x}^q}\left(\frac{\partial \bar{x}^p}{\partial x^i}\right)$$

$$= \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial A^i}{\partial x^j}\cdot\frac{\partial x^j}{\partial \bar{x}^q} + A^i \frac{\partial}{\partial x^j}\left(\frac{\partial \bar{x}^p}{\partial x^i}\right)\frac{\partial x^j}{\partial \bar{x}^q}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^q}\frac{\partial A^i}{\partial x^j} + \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^i}\cdot\frac{\partial x^j}{\partial \bar{x}^q} A^i \longrightarrow (2)$$

Due to the presence of the 2nd term on R.H.S of equation (2) $\frac{\partial A^i}{\partial x^j}$ does not transform according to the transformation law of the Contravariant vector. Hence $\frac{\partial A^i}{\partial x^j}$ is not

a tensor.

Theorem# If A_i is a covariant vector, show that the partial derivatives $\frac{\partial A_i}{\partial x^j}$ defined in each co-ordinate system, transform according to the rule

$$\frac{\partial \bar{A}_p}{\partial \bar{x}^q} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial A_j}{\partial x^j} + \frac{\partial^2 x^i}{\partial \bar{x}^p \partial \bar{x}^q} A_i$$

Proof# Let the co-ordinate transformation be

$$\bar{x}^j = \bar{x}^j (x^1, x^2, \ldots x^n)$$
$$j = 1, 2, \ldots n$$

Let $\bar{A}_p$ be the corresponding components of the covariant vector in the co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$, then Laws of transformation is

$$\bar{A}_p = \frac{\partial x^i}{\partial \bar{x}^p} A_i \longrightarrow (1)$$

Differentiating both sides of equation (1) w.r.t $\bar{x}^q$

$$\frac{\partial \bar{A}_p}{\partial \bar{x}^p} = \frac{\partial}{\partial \bar{x}^q}\left(\frac{\partial x^i}{\partial \bar{x}^p} A_i\right)$$

$$= \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial A_i}{\partial \bar{x}^q} + A_i \frac{\partial}{\partial \bar{x}^q}\left(\frac{\partial x^i}{\partial \bar{x}^p}\right)$$

$$= \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial A_i}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^q} + A_i \frac{\partial^2 x^i}{\partial \bar{x}^q \partial \bar{x}^p} \rightarrow (2)$$

Due to the presence of the 2nd term on R.H.S of equation (2) $\frac{\partial A_i}{\partial x^j}$ does not transform according to the transformation law of the Contravariant vector. Hence $\frac{\partial A_i}{\partial x^j}$ is not a tensor

Theorem# Prove that if all the components of tensor are zero in one Co-ordinate system they are zero in all Co-ordinate systems.

Proof# Let $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ be the Components of a tensor of indicated type and order in the Co-ordinate system $(x^1, x^2, \cdots x^n)$

Let $\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s}$ be the corresponding Components in the Co-ordinate system $(\bar{x}^1, \bar{x}^2, \cdots \bar{x}^n)$, then transformation law is

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \cdot A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$$

If $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = 0$, then $\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} = 0$

which proves the Theorem

Addition#

The sum of two or more tensors of the same rank and type (i.e the same number of Contravariant and the same no of Co-variant indices) is also a tensor of the same rank and type. Addition of tensors is Commutative and associative.

Theorem# Let $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ and $B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ be two mixed tensors of Contravariant order r and Covariant order s, then prove that their sum

$A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} + B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is also a mixed tensor of Contravariant order r and Covariant order s.

Proof# Let $C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} + B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$

Then we have to prove that $C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$

is also a mixed tensor of Contravariant order r and Covariant order s. Let

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} \quad \text{and} \quad \bar{B}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s}$$

be the corresponding components of the given tensors in the system $(\bar{x}^1, \bar{x}^2 \cdots \bar{x}^n)$, then their transformation laws are

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \times A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \longrightarrow (1)$$

and

$$\bar{B}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} B^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \longrightarrow (2)$$

Adding (1) and (2)

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s} + \bar{B}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s}$$

$$= \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \left[A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} + B^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} \right]$$

$$\text{or } \bar{C}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$$

$\Rightarrow C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is a mixed tensor of Contravariant order r and Covariant order s.

By. M. Hussain Lecturer (Maths) Asghar Mall College

Subtraction

The difference of two tensors of the same rank and type is also a tensor of the same rank and type.

As in case of addition we can similarly prove that $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ and $B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ are two mixed tensors of Contravariant order r and Co-variant order s, then the difference $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} - B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is also a mixed tensor of Contravariant order r and Covariant order s

By Muhammad Hussain Lecturer (Maths) Govt College Asghar Mall.

Example # If A_k^{ij} and B_k^{ij} are two tensors of order 3, then prove that

(i) $C_k^{ij} = A_k^{ij} + B_k^{ij}$

(ii) $D_k^{ij} = A_k^{ij} - B_k^{ij}$ are also tensors of the same type and order

Sol # Let $\bar{A}_\ell^{pq}$ and $\bar{B}_\ell^{pq}$ be the corresponding components of the given tensors in the co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$, then the Laws of transformation are

$$\bar{A}_\ell^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \cdot \frac{\partial x^k}{\partial \bar{x}^\ell} A_k^{ij} \longrightarrow (1)$$

$$\bar{B}_\ell^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \cdot \frac{\partial x^k}{\partial \bar{x}^\ell} B_k^{ij} \longrightarrow (2)$$

Adding (1) and (2) we get

$$\bar{A}_\ell^{pq} + \bar{B}_\ell^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \cdot \frac{\partial x^k}{\partial \bar{x}^\ell} (A_k^{ij} + B_k^{ij})$$

$$\bar{C}_\ell^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \cdot \frac{\partial x^k}{\partial \bar{x}^\ell} C_k^{ij} \longrightarrow (2)$$

where $\bar{C}_k^{ij} = \bar{A}_k^{ij} + \bar{B}_k^{ij}$

(3) shows that C_k^{ij} is a tensor of order 3 i.e Contravariant of order 2 and Co-variant of order 1

(ii) Subtracting equation (2) from (1) we

$$\bar{A}^{pq}_{\ell} - \bar{B}^{pq}_{\ell} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \cdot \frac{\partial x^h}{\partial \bar{x}^\ell} (A^{ij}_h - B^{ij}_h)$$

$$\Rightarrow \bar{D}^{pq}_{\ell} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial \bar{x}^q}{\partial x^j} \cdot \frac{\partial x^h}{\partial \bar{x}^\ell} D^{ij}_h$$

where $\bar{D}^{ij}_h = \bar{A}^{ij}_h - \bar{B}^{ij}_h$

$\Rightarrow D^{ij}_h$ is a tensor of order 3

Equal Tensors

Two tensors of the same rank and type are said to be equal in a certain Co-ordinate system if their corresponding components are equal.

Theorem # Prove that if the components of two tensors are equal in one Co-ordinate System, then they are equal in all Co-ordinates system.

Proof # let $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ and $B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$

be components of two equal tensors in a Co-ordinate system $(x^1, x^2 \ldots x^n)$, then

$$A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$$

$\Rightarrow$

$$A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} - B^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = 0 \longrightarrow (1)$$

in the Co-ordinate system $(x^1, x^2 \ldots x^n)$ We know that if all the components of a tensor are zero in one Co-ordinate systems, they are zero in all Co-ordinate systems. It follows that difference in (1) must be zero in a every other Co-ordinate System. Hence the tensors are equal in all Co-ordinate systems. This proves the theorem

Multiplication of a tensor by a Scalar#

The multiplication of a tensor of any rank by a scalar yields another tensor of the same rank.

Theorem# Prove that if λ is a scalar and $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is a mixed tensor of Contravariant order r and Covariant order s, then $\lambda A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is also a mixed tensor of the same order

Proof # Let $C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} = \lambda A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$

Let $\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$ be the corresponding components of the given tensor in the co-ordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n)$, then the law of transformation is

$$\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \cdot A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$$

Multiplying both sides of equation (ii) by the scalar λ, we get

$$\lambda \bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \left(\lambda A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \right)$$

$$\bar{C}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$$

This shows that $C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is also a mixed tensor of contravariant order r and covariant order s.

Linear Combination of Tensors

If $A_1, A_2 \ldots A_n$ are are tensors of the same type and order and if $\lambda_1, \lambda_2 \ldots \lambda_n$ are invariant or scalars, then the linear combination of these tensors $\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_n A_n$ is a tensor of the same type and order

Example # Show that if λ & μ are invariant and A^i, B^i are Components of Contravariant Vectors, then the vector defined in all Co-ordinates system by $\lambda A^i + \mu B^i$ is a Centravariant vector.

Sol # Since λ & μ are invariant

$$\therefore \quad \bar{\lambda} = \lambda \qquad \bar{\mu} = \mu$$

Let $\bar{A}^p$ and $\bar{B}^p$ be Corresponding Components of the given vectors in the Co-ordinate system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$, then transformation laws are

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \quad \& \quad \bar{B}^p = \frac{\partial \bar{x}^p}{\partial x^i} B^i$$

$$\bar{\lambda}\bar{A}^p + \bar{\mu}\bar{B}^p = \lambda \frac{\partial \bar{x}^p}{\partial x^i} A^i + \mu \frac{\partial \bar{x}^p}{\partial x^i} B^i$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} (\lambda A^i + \mu B^i)$$

Which shows that $\lambda A^i + \mu B^i$ is a

a Contravariant tensor of order one i.e Contravariant Vector.

By M. Hussain Lecturer (Math) Asghar Mall College

Outer Multiplication#

The outer product of two or more tensors is a tensor whose components are the products of the components of the given tensors.

Explaination# Let us suppose that we have two tensors A_i & B^i_k in 3-dimensional space. The outer product of two tensors is C^{ij}_{ik}, a tensor of 3rd order such that

$$C^{ij}_{ik} = A_i B^j_k$$

A_i stands for A_1, A_2, A_3

B^i_k stands for $B^1_1, B^1_2, B^1_3, B^2_1, B^2_2, B^2_3, B^3_1, B^3_2, B^3_3$

Taking products of components of A_i with those of B^i_k we have

$A_1 B^1_1, A_1 B^1_2, A_1 B^1_3, A_1 B^2_1, A_1 B^2_2, A_1 B^2_3, A_1 B^3_1, A_1 B^3_2, A_1 B^3_3$

$A_2 B^1_1, A_2 B^1_2, A_2 B^1_3, A_2 B^2_1, A_2 B^2_2, A_2 B^2_3, A_1 B^3_1, A_1 B^3_2, A_1 B^3_3$

$A_3 B_1^1$, $A_3 B_2^1$, $A_3 B_3^1$, $A_3 B_1^2$, $A_3 B_2^2$, $A_3 B_3^2$
$A_3 B_1^3$, $A_3 B_2^3$, $A_3 B_3^3$

This new group of functions form a tensor of the type C^{j}_{ik}

Theorem# If $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ and $B^{h_1 h_2 \cdots h_r}_{k_1 k_2 \cdots k_s}$ are two mixed tensors of ranks $(p+q)$ and $(r+s)$ respectively, then prove that the outer product of these tensors is also a tensor of the same type and order $p+q+r+s$, Contravariant of order $p+r$ and Covariant of order $q+s$

Proof# Let $C^{i_1 i_2 \cdots i_p h_1 h_2 \cdots h_r}_{j_1 j_2 \cdots j_q k_1 k_2 \cdots k_s}$

$$= A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} \cdot B^{h_1 h_2 \cdots h_r}_{k_1 k_2 \cdots k_s} \longrightarrow (1)$$

Let $\bar{A}^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q}$ & $\bar{B}^{\gamma_1 \gamma_2 \cdots \gamma_r}_{\delta_1 \delta_2 \cdots \delta_s}$ be

be the Corresponding Components of the given tensors in the Co-ordinates system $(\bar{x}^1, \bar{x}^2, \cdots \bar{x}^n)$, then their transformation laws are

$$\bar{A}^{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_p}_{\bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_q} = \frac{\partial \bar{x}^{\bar{\alpha}_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{\bar{\alpha}_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{\bar{\alpha}_p}}{\partial x^{i_p}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{\bar{\beta}_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{\bar{\beta}_2}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{\bar{\beta}_q}} A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} \longrightarrow (2)$$

$$\bar{B}^{\bar{\gamma}_1\bar{\gamma}_2\cdots\bar{\gamma}_r}_{\bar{\delta}_1\bar{\delta}_2\cdots\bar{\delta}_s} = \frac{\partial \bar{x}^{\bar{\gamma}_1}}{\partial x^{k_1}} \cdot \frac{\partial \bar{x}^{\bar{\gamma}_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{\bar{\gamma}_r}}{\partial x^{k_r}} \cdot \frac{\partial x^{l_1}}{\partial \bar{x}^{\bar{\delta}_1}} \cdot \frac{\partial x^{l_2}}{\partial \bar{x}^{\bar{\delta}_2}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{\bar{\delta}_s}} \cdot B^{k_1 k_2 \cdots k_r}_{l_1 l_2 \cdots l_s} \longrightarrow (3)$$

Multiplying (2) & (3) we get.

$$\bar{A}^{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_p}_{\bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_q} \cdot \bar{B}^{\bar{\gamma}_1\bar{\gamma}_2\cdots\bar{\gamma}_r}_{\bar{\delta}_1\bar{\delta}_2\cdots\bar{\delta}_s}$$

$$= \frac{\partial \bar{x}^{\bar{\alpha}_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{\bar{\alpha}_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{\bar{\alpha}_p}}{\partial x^{i_p}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{\bar{\beta}_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{\bar{\beta}_2}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{\bar{\beta}_q}} \cdots$$

$$\frac{\partial \bar{x}^{\bar{\gamma}_1}}{\partial x^{k_1}} \cdot \frac{\partial \bar{x}^{\bar{\gamma}_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{\bar{\gamma}_r}}{\partial x^{k_r}} \cdot \frac{\partial x^{l_1}}{\partial \bar{x}^{\bar{\delta}_1}} \cdot \frac{\partial x^{l_2}}{\partial \bar{x}^{\bar{\delta}_2}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{\bar{\delta}_s}} \left(A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} \cdot B^{k_1 k_2 \cdots k_r}_{l_1 l_2 \cdots l_s} \right)$$

$$\bar{C}^{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_p\,\bar{\gamma}_1\bar{\gamma}_2\cdots\bar{\gamma}_r}_{\bar{\beta}_1\bar{\beta}_2\cdots\bar{\beta}_q\,\bar{\delta}_1\bar{\delta}_2\cdots\bar{\delta}_s} = \frac{\partial \bar{x}^{\bar{\alpha}_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{\bar{\alpha}_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{\bar{\alpha}_p}}{\partial x^{i_p}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{\bar{\beta}_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{\bar{\beta}_2}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{\bar{\beta}_q}} \cdot \frac{\partial \bar{x}^{\bar{\gamma}_1}}{\partial x^{k_1}} \cdot \frac{\partial \bar{x}^{\bar{\gamma}_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{\bar{\gamma}_r}}{\partial x^{k_r}} \cdot \frac{\partial x^{l_1}}{\partial \bar{x}^{\bar{\delta}_1}} \cdot \frac{\partial x^{l_2}}{\partial \bar{x}^{\bar{\delta}_2}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{\bar{\delta}_s}}$$

$$\times C^{i_1 i_2 \cdots i_p\, k_1 k_2 \cdots k_r}_{j_1 j_2 \cdots j_q\, l_1 l_2 \cdots l_s}$$

This shows that the outer product of tensors of orders $(p+q)$ & $(r+s)$ is again a tensor of contravariant order $p+r$ and covariant order $q+s$.

Question# Show that if A^i_j and B_k are two tensors, then their outer product $A^i_j B_k$ is a tensor of order Three

Sol# Let $C^i_{jk} = A^i_j B_k$. Let $\bar{A}^p_q$ and $\bar{B}_r$ be the corresponding components of the given tensors in the Co-ordinate system. Then the Laws of transformation are

$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i_j \longrightarrow (1)$$

$$\bar{B}_r = \frac{\partial x^k}{\partial \bar{x}^r} B_k \longrightarrow (2)$$

Multiplying these equations

$$\bar{A}^p_q \bar{B}_r = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial x^k}{\partial \bar{x}^r} A^i_j B_k$$

$$\bar{C}^p_{qr} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial x^k}{\partial \bar{x}^r} C^i_{jk}.$$

which shows that $C^{i}_{jk} = A^{i}_{j} B_k$ is a mixed tensor of order 3 i.e Contravariant order 1 and Covariant order 2.

Contraction

The process of placing a Contravariant index equal to a Covariant index in a tensor and summing over the equal indices is called Contraction.

By this process the rank of a tensor is lowered by 2 (or by any even number).

Note General tensor having only Contravariant indices or only Covariant indices) Can not be Contracted.

Contraction Theorem

If $A^{i_1 i_2 \ldots i_m \ldots i_r}_{j_1 j_2 \ldots j_n \ldots j_s}$ is a mixed tensor of order $r+s$, (Contravariant of order r and Covariant of order s), then prove that the Contraction of this tensor w.r.t indices i_m & j_n is a tensor of Rank $r+s-2$

Proof # Let $\bar{A}^{p_1 p_2 \ldots p_m \ldots p_r}_{q_1 q_2 \ldots q_n \ldots q_s}$ be

the Corresponding Components of the given tensor in the System $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots \bar{x}^N)$

, then the transformation laws is

$$\bar{A}^{p_1 p_2 \cdots p_m \cdots p_r}_{q_1 q_2 \cdots q_n \cdots q_s} = \left(\frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}}\right)$$

$$\left(\frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}}\right) A^{i_1 i_2 \cdots i_m \cdots i_r}_{j_1 j_2 \cdots j_n \cdots j_s}$$

Let us contract w.r.t indices i_m & j_n
Then we must show that the resulting tensor $A^{i_1 i_2 \cdots i_m \cdots i_r}_{j_1 j_2 \cdots i_m \cdots j_s}$ is order $r+s-2$

Equating the corresponding indices p_m & q_n and summing we get

$$\bar{A}^{p_1 p_2 \cdots p_m \cdots p_r}_{q_1 q_2 \cdots p_m \cdots q_s} = \left(\frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_m}}{\partial x^{i_m}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}}\right)$$

$$\left(\frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_n}}{\partial \bar{x}^{p_m}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}}\right) A^{i_1 i_2 \cdots i_m \cdots i_r}_{j_1 j_2 \cdots j_n \cdots j_s}$$

$$= \frac{\partial x^{j_n}}{\partial \bar{x}^{p_m}} \frac{\partial \bar{x}^{p_m}}{\partial x^{i_m}} \underbrace{\left(\frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}}\right)\left(\frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots\right.}_{(r-1) \text{ terms}}$$

$$\left. \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}}\right) A^{i_1 i_2 \cdots i_m \cdots i_r}_{j_1 j_2 \cdots j_n \cdots j_s}$$

$$= \delta^{j_n}_{i_m} \left(\frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}}\right)\left(\frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}}\right)$$

$$A^{i_1 i_2 \cdots i_m \cdots i_r}_{j_1 j_2 \cdots j_n \cdots j_s}$$

$$= \left(\frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}}\right)\left(\frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}}\right)$$

$(r-1)$ terms $\qquad$ $(s-1)$ terms

$$A^{i_1 i_2 \cdots i_m \, i_r}_{j_1 j_2 \cdots i_m \cdots j_s}$$

$$\Rightarrow A^{i_1 i_2 \cdots i_m \cdots i_r}_{j_1 j_2 \cdots i_m \cdots j_s} \text{ is a tensor}$$

of rank $r-1+s+1 = r+s-2$

Question# Show that any contraction of the tensor A^i_{jk} results in a covariant tensor.

Sol# Let $\bar{A}^{p}_{q_1 q_2}$ be the components in a co-ordinate system $(\bar{x}^1, \bar{x}^2 \cdots \bar{x}^n)$

Then law of transformation is

$$\bar{A}^{p}_{q_1 q_2} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^k}{\partial \bar{x}^{q_2}} A^i_{jk}$$

We may Contract on either $i=j$ or $i=k$

Let us Contract wrt indices i & j. We must show that A^i_{ik} is a Covariant vector. Place the Corresponding indices p & q equal to each other and sum over this index. Then

$$\bar{A}^{p}_{pr} = \frac{\partial \bar{x}^{p}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{p}} \cdot \frac{\partial x^{k}}{\partial \bar{x}^{r}} A^{i}_{jk}$$

$$= \frac{\partial x^{j}}{\partial \bar{x}^{p}} \cdot \frac{\partial \bar{x}^{p}}{\partial x^{i}} \cdot \frac{\partial x^{k}}{\partial \bar{x}^{r}} A^{i}_{jk}$$

$$= \delta^{j}_{i} \frac{\partial x^{k}}{\partial \bar{x}^{r}} A^{i}_{jk} = \frac{\partial x^{k}}{\partial \bar{x}^{r}} A^{i}_{ik}$$

which shows that A^{i}_{ik} is a Covariant vector.

Contracting w.r.t. indices i & k. place the corresponding indices p & r equal to each other and sum over this index. Then

$$\bar{A}^{p}_{qp} = \frac{\partial \bar{x}^{p}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{q}} \frac{\partial x^{k}}{\partial \bar{x}^{p}} A^{i}_{jk}$$

$$= \delta^{k}_{i} \frac{\partial x^{j}}{\partial \bar{x}^{q}} A^{i}_{jk} = \frac{\partial x^{j}}{\partial \bar{x}^{q}} A^{i}_{ji}$$

$$\bar{C}_{q} = \frac{\partial x^{j}}{\partial \bar{x}^{q}} C_{j}$$

which shows that $C_j = A^{i}_{ji}$ is a Covariant tensor of order one or Covariant vector. In either case, the transformation law is that of a Covariant vector.

By Muhammad Hussain Lecturer (Maths) Govt. College Asghar Mall.

Question # Prove that the Contraction of tensor A^i_j is a scalar or invariant

Sol # we have

$$\bar{A}^{\bar{p}}_{\bar{q}} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i_j$$

putting $p = q$ and summing

$$\bar{A}^{\bar{p}}_{p} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^p} A^i_j$$

$$= \delta^j_i A^i_j = A^i_i$$

Then $\bar{A}^p_p = A^i_i$ and it follows that A^i_i must be an invariant

Inner Multiplication

The process of multiplying tensors (outer multiplication) and then Contracting is called the inner multiplication and the result is called an inner product of the given tensors.

Theorem # If $A^{i_1 i_2 \cdots i_m \cdots i_p}_{j_1 j_2 \cdots \cdots j_q}$ and $B^{k_1 k_2 \cdots k_r}_{l_1 l_2 \cdots l_n \cdots l_s}$ are two mixed tensors of rank $(p+q)$ and $(r+s)$ respectively, prove that the inner product of these

tensors is a tensor of rank $p+q+r+s-2$.

Proof #

Let $C^{i_1 i_2 \ldots i_m \ldots i_p k_1 k_2 \ldots k_q}_{j_1 j_2 \ldots \ldots j_q l_1 l_2 \ldots l_n \ldots l_s} = A^{i_1 i_2 \ldots i_m \ldots i_p k_1 k_2 \ldots k_q}_{j_1 j_2 \ldots \ldots j_q} \cdot B_{l_1 l_2 \ldots l_n \ldots l_s}$ $\longrightarrow$ (1)

and let $\bar{A}^{\alpha_1 \alpha_2 \ldots \alpha_m \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \ldots \beta_q}$ and $\bar{B}^{\gamma_1 \gamma_2 \ldots \gamma_r}_{\delta_1 \delta_2 \ldots \delta_n \ldots \delta_s}$

be the corresponding components of given tensors in the system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$. Then the transformation laws are

$$\bar{A}^{\alpha_1 \alpha_2 \ldots \alpha_m \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \ldots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{\alpha_2}}{\partial x^{i_2}} \ldots \frac{\partial \bar{x}^{\alpha_m}}{\partial x^{i_m}} \ldots \frac{\partial x^{i_p}}{\partial \bar{x}^{\beta_1}} \frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_2}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{\beta_q}} \ldots \frac{\partial x^{j_q}}{\partial \bar{x}^{\beta_q}} A^{i_1 i_2 \ldots i_m \ldots i_p}_{j_1 j_2 \ldots \ldots j_q}$$ $\longrightarrow$ (2)

$$\bar{B}^{\gamma_1 \gamma_2 \ldots \ldots \gamma_r}_{\delta_1 \delta_2 \ldots \delta_n \ldots \delta_s} = \frac{\partial \bar{x}^{\gamma_1}}{\partial x^{k_1}} \cdot \frac{\partial \bar{x}^{\gamma_2}}{\partial x^{k_2}} \ldots \frac{\partial \bar{x}^{\gamma_r}}{\partial x^{k_r}} \cdot \frac{\partial x^{l_1}}{\partial \bar{x}^{\delta_1}} \ldots \frac{\partial x^{l_s}}{\partial \bar{x}^{\delta_s}} B^{k_1 k_2 \ldots \ldots k_q}_{l_1 l_2 \ldots l_n \ldots l_s}$$ $\longrightarrow$ (3)

Multiplying equations (2) & (3)

$$\bar{A}^{\alpha_1 \alpha_2 \ldots \alpha_m \ldots \alpha_p}_{\beta_1 \beta_2 \ldots \ldots \beta_q} \cdot \bar{B}^{\gamma_1 \gamma_2 \ldots \ldots \gamma_r}_{\delta_1 \delta_2 \ldots \delta_n \ldots \delta_s}$$

$$= \left(\frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{\alpha_2}}{\partial x^{i_2}} \ldots \frac{\partial \bar{x}^{\alpha_m}}{\partial x^{i_m}} \ldots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{\beta_2}} \ldots \frac{\partial x^{j_q}}{\partial \bar{x}^{\beta_q}} \right) \times$$

$$\left(\frac{\partial \bar{x}^{r_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{r_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{r_2}}{\partial x^{k_2}} \cdot \frac{\partial x^{l_1}}{\partial \bar{x}^{\delta_1}} \frac{\partial x^{l_2}}{\partial \bar{x}^{\delta_2}} \cdots \frac{\partial x^{l_n}}{\partial \bar{x}^{\delta_n}} \cdot \frac{\partial x^{l_s}}{\partial \bar{x}^{\delta_s}}\right)$$

$$A^{i_1 i_2 \ldots i_m \ldots i_p}_{j_1 j_2 \ldots \ldots j_q} \cdot B^{k_1 k_2 \ldots \ldots k_2}_{l_1 l_2 \ldots l_n \ldots l_s}$$

$$\bar{C}^{\alpha_1 \alpha_2 \ldots \alpha_m \ldots \alpha_p \gamma_1 \gamma_2 \ldots \gamma_2}_{\beta_1 \beta_2 \ldots \beta_q \delta_1 \delta_2 \ldots \delta_n \ldots \delta_s} = \left(\frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{\alpha_2}}{\partial \bar{x}^{i_2}} \cdot \right.$$

$$\left. \cdots \frac{\partial \bar{x}^{\alpha_m}}{\partial x^{i_m}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \cdot \frac{\partial \bar{x}^{\gamma_1}}{\partial x^{k_1}} \cdot \frac{\partial \bar{x}^{\gamma_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{\gamma_2}}{\partial x^{k_2}}\right) \times$$

$$\left(\frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{\beta_2}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{\beta_q}} \cdot \frac{\partial x^{l_1}}{\partial \bar{x}^{\delta_1}} \cdots \frac{\partial x^{l_n}}{\partial \bar{x}^{\delta_n}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{\delta_s}}\right) C$$

$$\times C^{i_1 i_2 \ldots i_m \ldots i_p \, k_1 k_2 \ldots k_2}_{j_1 j_2 \ldots j_q \, l_1 l_2 \ldots l_n \ldots l_s} \longrightarrow (4)$$

which is a tensor of rank $p+2+q+s$

Let us contract w.r.t indices i_m & l_n. place the corresponding indices α_m and δ_n equal to each other in (4) we have.

$$\bar{C}^{\alpha_1 \alpha_2 \ldots \alpha_m \ldots \alpha_p \gamma_1 \gamma_2 \ldots \gamma_2}_{\beta_1 \beta_2 \ldots \beta_q \delta_1 \delta_2 \ldots \alpha_m \ldots \delta_s}$$

$$= \delta^{l_n}_{i_m} \left(\frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{\alpha_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}}\right)\left(\frac{\partial x^{l_1}}{\partial \bar{x}^{\delta_1}} \cdot \frac{\partial x^{l_2}}{\partial \bar{x}^{\delta_2}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{\delta_s}}\right)$$

$$\times \left(\frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{\beta_2}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{\beta_q}}\right)\left(\frac{\partial x^{l_1}}{\partial \bar{x}^{\delta_1}} \cdot \frac{\partial x^{l_2}}{\partial \bar{x}^{\delta_2}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{\delta_s}}\right)$$

$$C^{i_1 i_2 \ldots \ldots i_m \ldots i_p \, k_1 k_2 \ldots k_2}_{j_1 j_2 \ldots \ldots j_q \, l_1 l_2 \ldots l_m \ldots l_s}$$

$$= \left(\frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{\alpha_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}}\right)\left(\frac{\partial \bar{x}^{\eta_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{\eta_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{\eta_m}}{\partial x^{k_2}}\right)$$

$$\left(\frac{\partial x^{j_1}}{\partial \bar{x}^{\beta_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{\beta_2}} \cdots \frac{\partial x^{j_2}}{\partial \bar{x}^{\beta_2}}\right)\left(\frac{\partial x^{l_1}}{\partial \bar{x}^{\delta_1}} \cdot \frac{\partial x^{l_2}}{\partial \bar{x}^{\delta_2}} \cdots \frac{\partial x^{l_s}}{\partial \bar{x}^{\delta_s}}\right)$$

$$C^{i_1 i_2 \cdots i_m \cdots i_p k_1 k_2 \cdots k_2}_{j_1 j_2 \cdots j_q \; l_1 l_2 \cdots i_m \cdots l_s} \longrightarrow (5)$$

which shows that $C^{i_1 i_2 \cdots i_p \; k_1 k_2 \cdots k_2}_{j_1 j_2 \cdots j_q \; l_1 l_2 \cdots l_s}$

is a tensor of order $p-1+q+2+s-1$

$= p+q+2+s-2$

By M. Hussain Lecturer (Maths) Asghar Mall College

Question # Show that any inner product of tensors A^i_j and B^{kl}_m is a tensor of rank 3

Sol # $\because A^i_j$ & B^{kl}_m are tensors, Then

$$\bar{A}^p_q = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i_j \longrightarrow (1)$$

$$\bar{B}^{rs}_t = \frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial \bar{x}^s}{\partial x^l} \cdot \frac{\partial x^m}{\partial \bar{x}^t} B^{kl}_m \longrightarrow (2)$$

Multiplying (1) & (2)

$$\bar{A}^p_q \bar{B}^{rs}_t = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial \bar{x}^r}{\partial x^k} \cdot \frac{\partial \bar{x}^s}{\partial x^l} \cdot \frac{\partial x^m}{\partial \bar{x}^t} A^i_j B^{kl}_m$$

$\longrightarrow (3)$

Let us Contract w.r.t indices i and m in place the Corresponding indices p and t equal to each other in equation (3)

$$\bar{A}^{p}_{q}\,\bar{B}^{rs}_{p} = \frac{\partial \bar{x}^{p}}{\partial x^{i}}\frac{\partial x^{j}}{\partial \bar{x}^{q}}\frac{\partial x^{k}}{\partial \bar{x}^{r}}\frac{\partial \bar{x}^{s}}{\partial x^{l}}\frac{\partial x^{m}}{\partial \bar{x}^{p}}\,A^{i}_{j}\,B^{kl}_{m}$$

$$= \delta^{m}_{i}\,\frac{\partial x^{j}}{\partial \bar{x}^{q}}\frac{\partial \bar{x}^{r}}{\partial x^{k}}\cdot\frac{\partial \bar{x}^{s}}{\partial x^{l}}\,A^{i}_{j}\,B^{kl}_{m}$$

$$= \frac{\partial x^{j}}{\partial \bar{x}^{q}}\frac{\partial \bar{x}^{r}}{\partial x^{k}}\frac{\partial \bar{x}^{s}}{\partial x^{l}}\,A^{i}_{j}\,B^{kl}_{i}$$

Shewing that $A^{i}_{j}\,B^{kl}_{i}$ is a tensor of rank 3

Note that by Contracting w.r.t k and j or l and j in the product $A^{i}_{j}\,B^{kl}_{m}$ we can similarly show that [illegible] product is a tensor of rank 3.

Alternate Method#

The outer product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. Then $A^{i}_{j}\,B^{kl}_{m}$ is a tensor of rank $3+2=5$. Since a Contraction lowers the rank by two, therefore it follows that any Contraction of $A^{i}_{j}\,B^{kl}_{m}$ is a tensor of rank $5-2=3$

Question# Show that the inner product of tensors A^i and B_j is an invariant

Sol# $\because$ A^i & B_j are tensors

$$\therefore \bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i$$

$$\bar{B}_q = \frac{\partial x^j}{\partial \bar{x}^q} B_j$$

$$\Rightarrow \bar{A}^p \bar{B}_q = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A^i B_j$$

By applying Contraction on p & q

$$\bar{A}^p \bar{B}_p = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^p} A^i B_j$$

$$= \delta^j_i A^i B_j = A^i B_i$$

So $A^i B_i$ is an invariant

Quotient Theorem#

If an inner product of a quantity X with an arbitrary tensor is itself a tensor, then X is also a tensor

Proof# Let B^k be an arbitrary Contravariant vector and $X = A^{i_1 i_2 \ldots i_s}_{j_1 j_2 \ldots j_s k}$ be an arbitrary quantity - Suppose

that $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s k} B^k = C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$

and $C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is a mixed tensor of order $r+s$. Then we are to prove that quantity X is also a tensor

In the Co-ordinate system $(x^1, x^2 \ldots x^n)$, we have

$$A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s k} B^k = C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \longrightarrow (1)$$

Let $\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s t}$, $\bar{B}^t$, $\bar{C}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$

be the components in the system $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n)$. Then

$$\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s t} \bar{B}^t = \bar{C}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} \longrightarrow (2)$$

Since B^k is an arbitrary Contravariant vector, therefore

$$\bar{B}^t = \frac{\partial \bar{x}^t}{\partial x^k} B^k \longrightarrow (3)$$

Also $C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is a mixed

tensor of order $r+s$, therefore

$$\bar{C}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} C^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s} \longrightarrow (4)$$

using (3) and (4) in (2) we get

$$\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s t} \frac{\partial \bar{x}^t}{\partial x^k} B^k = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s k} B^k$$

$\Rightarrow$

$$\left[\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s t} \frac{\partial \bar{x}^t}{\partial x^k} - \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} A^{i_1 i_2 \ldots i_r}_{j_1 \ldots j_s k} \right] \times B^k = 0$$

Since B^k is an arbitrary Contravariant vector, therefore

$$\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s t} \frac{\partial \bar{x}^t}{\partial x^k} - \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s k} = 0$$

$\Rightarrow$

$$\bar{A}^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s t} \frac{\partial \bar{x}^t}{\partial x^k} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \cdot A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s k}$$

Multiplying both sides by $\frac{\partial x^{k}}{\partial \bar{x}^{m}}$ we get

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s t} \frac{\partial \bar{x}^{t}}{\partial x^{k}} \frac{\partial x^{k}}{\partial \bar{x}^{m}} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \cdot \frac{\partial x^{k}}{\partial \bar{x}^{m}} \times A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s k}$$

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s t} \delta^{t}_{m} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdot \frac{\partial x^{j_2}}{\partial \bar{x}^{q_2}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \cdot \frac{\partial x^{k}}{\partial \bar{x}^{m}} A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s k}$$

$$\Rightarrow \bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s m} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \cdot \frac{\partial x^{k}}{\partial \bar{x}^{m}} A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s k}$$

Replacing the index m by t

$$\bar{A}^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s t} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \bar{x}^{q_1}} \cdots \frac{\partial x^{j_s}}{\partial \bar{x}^{q_s}} \cdot \frac{\partial x^{k}}{\partial \bar{x}^{t}} A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s k}$$

Which shows that $A^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s k}$ is a tensor of the indicated type and order

Question# A quantity A_{ij} is such that in the co-ordinate system $(x^1, x^2 \cdots x^n)$ $A_{ij} B^i = C_j$, where B^i is an arbitrary Contravariant vector and C_j is a (Contra) Covariant vector, then prove that A_{ij} is a Covariant tensor of order 2.

Solution# Given that

$$A_{ij} B^i = C_j \longrightarrow (1)$$

Let $\bar{A}_{pq}$, $\bar{B}^p$, $\bar{C}_q$ be corresponding components of the quantity, Contravariant vector and Covariant vector in $(\bar{x}^1, \bar{x}^2 \cdots \bar{x}^n)$, then

$$\bar{A}_{pq} \bar{B}^p = \bar{C}_q \longrightarrow (2)$$

Since B^i is an arbitrary Contravariant vector, therefore

$$\bar{B}^p = \frac{\partial \bar{x}^p}{\partial x^i} B^i \longrightarrow (3)$$

Also C_j is a Covariant vector

$$\bar{C}_q = \frac{\partial x^j}{\partial \bar{x}^q} C_j \longrightarrow (4)$$

putting (3) & (4) in (2)

$$\bar{A}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} B^i = \frac{\partial x^j}{\partial \bar{x}^q} C_j$$

$$= \frac{\partial x^j}{\partial \bar{x}^q} A_{ij} B^i \quad \text{by (1)}$$

$$\Rightarrow \left[\bar{A}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} - \frac{\partial x^j}{\partial \bar{x}^q} A_{ij} \right] B^i = 0$$

Since B^i is an arbitrary Contravariant vector, therefore

$$\bar{A}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} - \frac{\partial x^j}{\partial \bar{x}^q} A_{ij} = 0$$

or $$\bar{A}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} = \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$$

Multiplying both sides by $\frac{\partial x^i}{\partial \bar{x}^m}$ we get

$$\bar{A}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^i}{\partial \bar{x}^m} = \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$$

$$\Rightarrow \bar{A}_{pq} \delta^p_m = \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$$

$$\rightarrow \bar{A}_{mq} = \frac{\partial x^i}{\partial \bar{x}^m} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$$

Replacing m by p

$$\bar{A}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \cdot \frac{\partial x^j}{\partial \bar{x}^q} A_{ij}$$

which shows that A_{ij} a Covariant tensor of order 2

By M. Hussain Lecturer Govt. College Asghar Mall

Question# A quantity A_i is such That in the Co-ordinate system $(x^1, x^2 \ldots x^n)$ $A_i B^i$ is an invariant where B^i is an arbitrary Contravariant vector, then prove that A_i is a Covariant vector.

Sol# Let $\bar{A}_p$ and $\bar{B}^p$ be the Corresponding

Components of quantity and a Contravariant vector in the Co-ordinate system $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$. Then since $A_i B^i$ is an invariant we have

$$\bar{A}_p \bar{B}^p = A_i B^i \longrightarrow (1)$$

Since B^i is an arbitrary Contravariant vector, therefore

$$\bar{B}^p = \frac{\partial \bar{x}^p}{\partial x^i} B^i \longrightarrow (2)$$

using in (1)

$$\bar{A}_p \frac{\partial \bar{x}^p}{\partial x^i} B^i = A_i B^i$$

$$\Rightarrow \left[\bar{A}_p \frac{\partial \bar{x}^p}{\partial x^i} - A_i\right] B^i = 0$$

$\because$ B^i is an arbitrary

$\therefore$

$$\bar{A}_p \frac{\partial \bar{x}^p}{\partial x^i} - A_i = 0$$

$$\Rightarrow \bar{A}_p \frac{\partial \bar{x}^p}{\partial x^i} = A_i$$

Multiplying $\frac{\partial x^i}{\partial \bar{x}^q}$

$$\bar{A}_p \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} = \frac{\partial x^i}{\partial \bar{x}^q} A_i$$

$$\delta^p_q \bar{A}_p = \frac{\partial x^i}{\partial \bar{x}^q} A_i$$

$$\Rightarrow \bar{A}_q = \frac{\partial x^i}{\partial \bar{x}^q} A_i$$

$$\Rightarrow \bar{A}_p = \frac{\partial x^i}{\partial \bar{x}^p} A_i$$

which shows that A_i is a Covariant vector

Symmetric Tensor

A tensor is called Symmetric w.r.t two Contravariant (or Covariant) indices if the value of the Components of tensor remain unaltered, when these indices are interchanged.

e.g A_{ij} is symmetric w.r.t i, j if

$$A_{ij} = A_{ji}$$

and tensor B^{jkl} is symmetric w.r.t j & k if

$$B^{jkl} = B^{kjl}.$$

Symmetric Tensor

A tensor is called a symmetric tensor if it is symmetric w.r.t all pairs of Contravariant indices or all pairs of Covariant indices.

Skew - Symmetric or Anti-Symmetric Tensor.

A tensor is called skew - symmetric w.r.t two Contravariant or two Covariant indices if its components change sign when these indices are interchanged.

If a tensor is skew - symmetric w.r.t all pairs of Contravariant indices or all pairs

of Covariant indices, it is called Skew-symmetric

Theorem# Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric in a pair of Contravariant or Covariant indices.

Proof# We prove this result for the Contravariant tensor A^{ij}, we have

$$A^{ij} = \frac{1}{2}(A^{ij} + A^{ji}) + \frac{1}{2}(A^{ij} - A^{ji})$$
$$= B^{ij} + C^{ij}$$

But $B^{ij} = \frac{1}{2}(A^{ij} + A^{ji})$

$$= \frac{1}{2}(A^{ji} + A^{ij}) = B^{ji}$$

$\Rightarrow B^{ij}$ is symmetric

and $C^{ij} = \frac{1}{2}(A^{ij} - A^{ji})$

$$= -\frac{1}{2}(A^{ji} - A^{ij}) = -C^{ji}$$

is skew-symmetric

By Similar reasoning this result can be proved for any tensor.

By M. Husain Lectures (Math) Govt. College Asghar Mall

Theorem# Prove that in an n-dimensional space, a symmetric second order tensor A_{ij} has almost $n(n+1)/2$ different

Components.

Proof # Since A_{ij} is a covariant tensor of order 2, therefore it has following n^2 Components

$$A_{11} \quad A_{12} \quad A_{13} \quad \dots \quad A_{1n}$$
$$A_{21} \quad A_{22} \quad A_{23} \quad \dots \quad A_{2n}$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$A_{n1} \quad A_{n2} \quad A_{n3} \quad \dots \quad A_{nn}$$

The number of independent Components along the principal diagonal (i.e the number of terms with repeated indices) is n. So the number of Components with unequal indices are

$$n^2 - n$$

Since A_{ij} is symmetric, therefore

$A_{12} = A_{21}$ $A_{13} = A_{31}$ $\dots$ $A_{n-1,n} = A_{n,n-1}$

Thus the no of independent components of A_{ij} with unequal indices is $\frac{n^2-n}{2}$

Hence the maximum number of independent components of A_{ij} is

$$n + \frac{n^2-n}{2} = \frac{n(n+1)}{2}$$

By Muhammad Hussain Lecturer Govt College Asghar Mall RWP.

Theorem# Prove that in an-n-dimensional space, a skew-symmetric 2nd order tensor A_{ij} has almost $\frac{n(n-1)}{2}$ different components

Proof# Since A_{ij} is a covariant tensor of order 2, it has following n^2 components

$$A_{11} \quad A_{12} \quad \cdots \quad A_{1n}$$
$$A_{21} \quad A_{22} \quad \cdots \quad A_{2n}$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$A_{n1} \quad A_{n2} \quad \cdots \quad A_{nn}$$

Since A_{ij} is skew-symmetric, we have

$$A_{ij} = -A_{ji} \qquad i,j = 1,2,3\cdots n$$

Hence $A_{ii} = -A_{ii}$

$$\Rightarrow 2A_{ii} = 0 \quad \text{or} \quad A_{ii} = 0 \qquad \forall i = 1,2\cdots n$$

Thus $A_{11} = A_{22} = \cdots = A_{nn} = 0$

So the no of independent components on the leading diagonal is zero. Thus the no of distinct indices is $n^2 - n$. But due to skew-symmetry, this no is reduced to $\frac{n^2-n}{2}$

Hence no of independent components of A_{ij} is $0 + \frac{n^2-n}{2} = \frac{n(n-1)}{2}$

By Muhammad Hussain Lecturer (Maths) Govt. College Asghar Mall.

Invariance of Symmetric & Skew-Symmetric Properties of a Tensor#

Theorem# If a tensor is symmetric (skew-symmetric) w.r.t two contravariant indices or two covariant indices in one co-ordinate system, then is symmetric (skew-symmetric) w.r.t these indices in any other co-ordinate system.

Proof# Let $A^{i_1 i_2 \cdots i_r}$ be components of a tensor in co-ordinate system $(x^1, x^2, \cdots x^n)$. Let $\bar{A}^{p_1 p_2 \cdots p_r}$ be the components of this tensor in the co-ordinate system $(\bar{x}^1, \bar{x}^2 \cdots \bar{x}^n)$, then the law of transformation is

$$\bar{A}^{p_1 p_2 \cdots p_r} = \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdot \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} A^{i_1 i_2 \cdots i_r} \longrightarrow (1)$$

Suppose that this tensor is symmetric w.r.t Contravariant indices i_1, i_2. Then

$$A^{i_1 i_2 \cdots i_r} = A^{i_2 i_1 \cdots i_r}$$

From (1) we get

$$\bar{A}^{p_1 p_2 \cdots p_r} = \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \frac{\partial \bar{x}^{p_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{p_r}}{\partial x^{i_r}} A^{i_2 i_1 \cdots i_r}$$

$$= \bar{A}^{p_2 p_1 \cdots p_r}$$

Showing that the tensor $A^{i_1 i_2 \cdots i_r}$

remains symmetric in the Co-ordinate system $(x^1, x^2 \ldots x^n)$. Let the tensor $A^{i_1 i_2 \ldots i_2}$ in the Co-ordinate system $(x^1, x^2, \ldots x^n)$ is Skew-Symmetric w.r.t Contravariant indices i_1 & i_2. Then

$$A^{i_1 i_2 \ldots i_2} = -A^{i_2 i_1 \ldots i_2}$$

from ① we have

$$\bar{A}^{p_1 p_2 \ldots p_2} = -\frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} \cdot \frac{\partial \bar{x}^{p_1}}{\partial x^{i_2}} \ldots \frac{\partial \bar{x}^{p_2}}{\partial x^{i_2}} A^{i_2 i_1 \ldots i_2}$$

$$= -\bar{A}^{p_2 p_1 \ldots p_2}$$

Shewing that the tensor $A^{i_1 i_2 \ldots i_2}$ remains Skew-symmetric in the Co-ordinate system $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$. Similarly, we can shew that a tensor $A_{i_1 i_2 \ldots i_2}$ symmetric (Skew-Symmetric) w.r.t two covariant indices i_1 & i_2 in Co-ordinate system $(x^1, x^2, \ldots x^n)$ remains symmetric (Skew-symmetric) w.r.t these indices in the Co-ordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n)$

Note The symmetry and skew-symmetry of a general tensor can be defined provided the pairs of indices contain the same type of indices i.e either Contravariant or Covariant. Neither symmetry nor skew symmetry is normally defined for a pair of indices of which one is

is a contravariant and the other co-variant because in this case transformation to another co-ordinate system does not preserve symmetrical or skew-symmetrical property. An exception to this rule is kronecker delta tensor δ^i_j.

By M. Hussain Lectures (Math) Govt College Asghar Mall

Question# If a tensor A^{ijk}_{lm} is symmetric (skew-symmetric) w.r.t indices i & j in one co-ordinate system, show that it remains symmetric (skew-symmetric) w.r.t i and j in any co-ordinate system

Sol# Since only indices i & j are involved we shall prove the result for B^{ij}. If B^{ij} is symmetric, then

$$B^{ij} = B^{ji}$$

and $$\bar{B}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} B^{ij}$$

$$= \frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^p}{\partial x^i} B^{ji}$$

$$= \bar{B}^{qp}$$

and B^{ij} remains symmetric in the $(\bar{x}^1, \bar{x}^2 \dots \bar{x}^n)$ co-ordinate system

if B^{ij} is skew-symmetric i.e

$$B^{ij} = -B^{ji}$$

$$\bar{B}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} B^{ij}$$

$$= -\frac{\partial \bar{x}^q}{\partial x^j}\frac{\partial \bar{x}^p}{\partial x^i} B^{ji} = -\bar{B}^{qp}$$

and B^{ij} remains skew-symmetric in the $(\bar{x}^1, \bar{x}^2, \bar{x}^3, \ldots \bar{x}^n)$ Co-ordinate system

Theorem# If $\phi = a_{ij} A^i A^j$ Show that we can write $\phi = b_{ij} A^i A^j$ where b_{ij} Symmetric

Proof# Given that

$$\phi = a_{ij} A^i A^j = a_{ji} A^j A^i$$

$$\phi = a_{ji} A^i A^j$$

$$\Rightarrow 2\phi = a_{ij} A^i A^j + a_{ji} A^i A^j$$

$$= (a_{ij} + a_{ji}) A^i A^j$$

$$\phi = \frac{1}{2}(a_{ij} + a_{ji}) A^i A^j$$

$$= b_{ij} A^i A^j$$

Where $b_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) = b_{ji}$ is Symmetric

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Fundamental Property of Tensor Equation#

Theorem# if a tensor equation holds in one Co-ordinate system, then it holds in every Co-ordinate system i.e the form of a tensor equation remains the same in every Co-ordinate system.

Proof We prove this for a simple equation.

Let $A^i B_{jk} = 3 C^i_{jk}$

Let $\bar{A}^p \bar{B}_{qr}, \bar{C}^p_{qr}$ be the components in the system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n)$

We must prove that

$$\bar{A}^p \bar{B}_{qr} = 3 \bar{C}^p_{qr}$$

It is given that

$$A^i B_{jk} - 3 C^i_{jk} = 0 \longrightarrow (1)$$

$\because$ A^i, B_{jk} & C^i_{jk} are tensors

$\therefore$ Their laws of transformation are

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^i} A^i \longrightarrow (2)$$

$$\bar{B}_{qr} = \frac{\partial x^j}{\partial \bar{x}^q} \cdot \frac{\partial x^k}{\partial \bar{x}^r} B_{jk} \longrightarrow (3)$$

$$\bar{C}^p_{qr} = \frac{\partial \bar{x}^p}{\partial x^i} \cdot \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial x^k}{\partial \bar{x}^r} C^i_{jk} \longrightarrow (4)$$

using these we have.

$$\bar{A}^p \bar{B}_{qr} - 3 \bar{C}^p_{qr}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} A^i \frac{\partial x^j}{\partial \bar{x}^q} \cdot \frac{\partial x^k}{\partial \bar{x}^r} B_{jk} - 3 \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial x^k}{\partial \bar{x}^r} C^i_{jk}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial x^k}{\partial \bar{x}^r} (A^i B_{jk} - 3 C^i_{jk}) = 0$$

which shows that form of the tensor equation remains same in all Co-ordinate system

The Line Element and Metric Tensor.

We know that in 3-dimensional rectangular Co-ordinate system. (x, y, z), the square of the line element ds^2 is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \longrightarrow (1)$$

We transform (1) in a general curvilinear Co-ordinate system (u^1, u^2, u^3) in which position vector of the point is given by

$$\underline{r} = \underline{r}(u^1, u^2, u^3)$$

Then

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u^1} du^1 + \frac{\partial \vec{r}}{\partial u^2} du^2 + \frac{\partial \vec{r}}{\partial u^3} du^3$$

$$= \vec{a}_1 du^1 + \vec{a}_2 du^2 + \vec{a}_3 du^3$$

where $\vec{a}_i = \frac{\partial \vec{r}}{\partial u^i}$ are tangent vectors to the Co-ordinate curves u^i $(i = 1, 2, 3, 4)$

Now $ds^2 = d\vec{r} \cdot d\vec{r}$

$$= (\vec{\alpha}_1 du^1 + \vec{\alpha}_2 du^2 + \vec{\alpha}_3 du^3) \cdot (\vec{\alpha}_1 du^1 + \vec{\alpha}_2 du^2 + \vec{\alpha}_3 du^3)$$

$$= \vec{\alpha}_1 \cdot \vec{\alpha}_1 (du^1)^2 + \vec{\alpha}_1 \cdot \vec{\alpha}_2 du^1 du^2 + \vec{\alpha}_1 \cdot \vec{\alpha}_3 du^1 du^3 + \vec{\alpha}_2 \cdot \vec{\alpha}_1 du^2 du^1 + \vec{\alpha}_2 \cdot \vec{\alpha}_2 du^2 du^2 + \vec{\alpha}_2 \cdot \vec{\alpha}_3 du^2 du^3 + \vec{\alpha}_3 \cdot \vec{\alpha}_1 du^3 du^1 + \vec{\alpha}_3 \cdot \vec{\alpha}_2 du^3 du^2 + \vec{\alpha}_3 \cdot \vec{\alpha}_3 du^3 du^3$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \vec{\alpha}_i \cdot \vec{\alpha}_j \, du^i du^j$$

$$= \sum_i \sum_j g_{ij} \, du^i du^j \longrightarrow (1)$$

where $g_{ij} = \vec{\alpha}_i \cdot \vec{\alpha}_j$

$$= \frac{\partial \vec{r}}{\partial u^i} \cdot \frac{\partial \vec{r}}{\partial u^j}$$

Equation (1) is called the fundamental quadratic form or metric form. The quantities g_{ij} are called metric co-efficients. The space in which (1) is given is called 3-dimensional Euclidean space and is denoted by E^n

If the curvilinear co-ordinate system is orthogonal, then $g_{ij} = 0$ $i \neq j$

Generalising this concept, the square of the line element ds^2 in an n-

dimensional Co-ordinate system $(x^1, x^2 \ldots x^n)$ is given by the fundamental or metric form

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, dx^i dx^j \longrightarrow (2)$$

or using the Summation Convention

$$ds^2 = g_{ij}\, dx^i dx^j \quad (i,j = 1,2 \ldots n) \longrightarrow (3)$$

Equation (3) is called Riemannian metric and any space in which ds^2 is given by (3) is called n-dimensional Riemannian space and is denoted by R^n

In particular case of unit tangent vectors, we have

$$g_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Equation (3) takes the form

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^4)^2$$
$$= dx^i dx^i \longrightarrow (4)$$

(4) is called an Euclidian metric and such space is called an-n-dimensional space - denoted by E^n

IInd Approach

To find the Relative position of any point in n dimensional space w.r.t neighbouring point, the concept of metric (space) was introduced by Remann. He used the idea of 3-dim-al distance.

The space with Riemam metric is called Riemann space.

Let the co-ordinates of a point in n dimensional space be $(\xi^1, \xi^2 \ldots \xi^n)$ and those of neighbouring point be $(\xi^1+d\xi^1, \xi^2+d\xi^2, \ldots, \xi^n+d\xi^n)$

Suppose ds denotes the metric between the points. Then

$$(ds)^2 = d\xi^1 d\xi^1 + d\xi^2 d\xi^2 + \ldots + d\xi^n d\xi^n$$
$$= d\xi^i d\xi^i$$

This formula does not retain its form on co-ordinate transformation. e.g Suppose we are in X-system: $(x^1, x^2 \ldots x^n)$ which is changed to $\bar{X}$-system $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n)$. Then by transformation

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \ldots x^n)$$

or $$x^i = x^i(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n) \longrightarrow (1)$$

In X - system.

$$ds^2 = dx^i dx^i \longrightarrow (2)$$

under change of co-ordinates we have

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha$$

and

$$ds^2 = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha \cdot \frac{\partial x^i}{\partial \bar{x}^\beta} d\bar{x}^\beta$$

$$= \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^i}{\partial \bar{x}^\beta} d\bar{x}^\alpha d\bar{x}^\beta$$

$$= g_{\alpha\beta} d\bar{x}^\alpha d\bar{x}^\beta \longrightarrow (3)$$

where the co-efficients $g_{\alpha\beta}(\bar{x})$ are defined by

$$g_{\alpha\beta}(\bar{x}) = \frac{\partial x^i}{\partial \bar{x}^\alpha} \cdot \frac{\partial x^i}{\partial \bar{x}^\beta} \longrightarrow (4)$$

These co-efficients are symmetric w.r.t α, β

A space is said to be flat if the formula for $(ds)^2$ given by (2) retains its form whatever be the transformation of Co-ordinate system. In general it is not so but in some special cases it is possible. formula (3) is more general and and the space $(ds)^2$ changes according it under the transformation

of Co-ordinates is called Riemann space or Curved space.

There exists a Co-ordinate system $\bar{x}^1, \bar{x}^2 \ldots \bar{x}^n$ defined by

$$T: \quad \bar{x}^i = \bar{x}^i(x^1, x^2 \ldots x^n)$$

Such that under this transformation. $g_{\alpha\beta}$ reduce to Constants and by some further suitable transformation. $(ds)^2$ can be expressed as sum of n squares. It happens when Riemann Christoffel tensor vanishes. In this case formula ③ reduces to formula ② for ds^2.

It can be shown by quotient law or otherwise that function.

$$g_{\alpha\beta} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^i}{\partial \bar{x}^\beta}$$

form a symmetric tensor in Riemann space, of 2nd order Covariant in α and Co-variant in β

Clearly $(ds)^2 = g_{\alpha\beta}\, d\bar{x}^\alpha d\bar{x}^\beta$ is an invariant or tensor of zero order because $(ds)^2$ is an invariant. $d\bar{x}^\alpha, d\bar{x}^\beta$ are both Contraviant of 1st order. Hence by quotient law $g_{\alpha\beta}$ is covariant tensor

of order two.

It is symmetrical because an interchange of α & β does not change

$$\frac{\partial \bar{x}^i}{\partial \bar{x}^\alpha} \cdot \frac{\partial \bar{x}^i}{\partial \bar{x}^\beta}$$

Metric Tensor

The quantities g_{ij} are the components of covariant tensor of rank 2 and is called metric tensor or fundamental tensor

In matrix form the metric tensor is given by

$$\begin{bmatrix} g_{11} & g_{12} \cdots g_{1n} \\ g_{21} & g_{22} \cdots g_{2n} \\ \vdots & \\ g_{n1} & g_{n2} \cdots g_{nn} \end{bmatrix}$$

Theorem# if $ds^2 = g_{ij}\, dx^i dx^j$ is an invariant, show that g_{ij} is a symmetric covariant tensor of rank 2.

OR

Prove that g_{ij} is a symmetric covariant tensor of rank 2

Proof# We have

$$ds^2 = g_{ij}\, dx^i dx^j$$

Since ds^2 is an invariant

$$\bar{g}_{pq}\, d\bar{x}^p d\bar{x}^q = g_{ij}\, dx^i dx^j$$

$$= g_{ij} \frac{\partial x^i}{\partial \bar{x}^p} d\bar{x}^p \frac{\partial x^j}{\partial \bar{x}^q} d\bar{x}^q$$

$$= g_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} d\bar{x}^p d\bar{x}^q$$

Comparing we have

$$\bar{g}_{pq} = g_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q}$$

$\Rightarrow g_{ij}$ is a covariant tensor of rank 2

To shew its symmetry property we use the theorem that

if $\phi = a_{ij} A^i A^j$, then we can write

$\phi = b_{ij} A^i A^j$ where b_{ij} is a symmetric.

We note that

$\phi = ds^2$ $\quad A^i = dx^i$ and $A^j = dx^j$

It follows that g_{ij} can be chosen symmetric

OR

By definition

$$g_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial x^k}{\partial \bar{x}^j}$$ which is

not changed by changing i & j. so g_{ij} is symmetric

Note # Since the metric tensor is symmetric i.e $g_{ij} = g_{ji}$

Therefore no of independent Components of g_{ij} is at most $\frac{n(n+1)}{2}$

Theorem # Prove that for an orthogonal curvilinear Co-ordinate system

$$g_{12} = g_{21} = 0$$
$$g_{23} = g_{32} = 0$$
$$g_{31} = g_{13} = 0$$

Proof # By definition

$$g_{ij} = \frac{\partial \vec{r}}{\partial u^i} \cdot \frac{\partial \vec{r}}{\partial u^j}$$

When the curvilinear Co-ordinates are orthogonal, then angle between tangent vectors $\frac{\partial \vec{r}}{\partial u^i}$ & $\frac{\partial \vec{r}}{\partial u^j}$ is 90° and we have

$$g_{12} = \left|\frac{\partial \vec{r}}{\partial u^1}\right| \left|\frac{\partial \vec{r}}{\partial u^2}\right| \cos 90^\circ = 0 = g_{21}$$

Similarly $g_{13} = g_{31} = 0 = g_{23} = g_{32}$

Metric Tensor in Cylinderical and Spherical Polar Co-ordinates

We know that in Cartesian Co-ordinates the square of the differential of arc length

s is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \longrightarrow (1)$$

Cylinderical Polar Co-ordinates

In cylinder polar co-ordinates (r, θ, z) the line element is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

If $x^1 = r \quad x^2 = \theta \quad x^3 = z$

Then $ds^2 = (dx^1)^2 + r^2 (dx^2)^2 + (dx^3)^2 \longrightarrow (2)$

Also

$$ds^2 = g_{ij}\, dx^i dx^j$$

$$= g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{12}\,dx^1 dx^2 + g_{13}\,dx^1 dx^3 + g_{21}\,dx^2 dx^1 + g_{22}\,dx^2 dx^2 + g_{23}\,dx^2 dx^3 + g_{31}\,dx^3 dx^1 + g_{32}\,dx^3 dx^2 + g_{33}\,dx^3 dx^3 \longrightarrow (3)$$

Comparing (2) & (3)

$g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = 1$

$g_{12} = g_{21} = 0 \qquad g_{23} = g_{32} = 0 \qquad g_{13} = g_{31} = 0$

In matrix form, the metric tensor can be written as

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By Muhammad Hussain Lecturer (Maths) Govt. College Asghar Mall.

OR

In, cylindrical polar co-ordinates

$\underline{r} = \underline{r}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$

and

$x^1 = x = r\cos\theta = \bar{x}^1 \cos\bar{x}^2$ $\quad x^1 = x$

$x^2 \quad y = r\sin\theta = \bar{x}^1 \sin\bar{x}^2$ $\quad x^2 = y$

$x^3 = z = z \quad , \quad = \quad , \quad \bar{x}^3$ $\quad x^3 = z$

$$g_{ij} = \frac{\partial \underline{r}}{\partial x^i} \cdot \frac{\partial \underline{r}}{\partial x^j}$$

$$g_{12} = \frac{\partial \underline{r}}{\partial x^1} \cdot \frac{\partial \underline{r}}{\partial x^2}$$

OR

$$g_{ij} = \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\alpha}{\partial \bar{x}^j} \qquad \alpha = 1, 2, 3$$

$$g_{12} = \frac{\partial x^\alpha}{\partial \bar{x}^1} \frac{\partial x^\alpha}{\partial \bar{x}^2}$$

$$= \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^1}{\partial \bar{x}^2} + \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^2} + \frac{\partial x^3}{\partial \bar{x}^1} \frac{\partial x^3}{\partial \bar{x}^2}$$

$$= \cos\bar{x}^2 (-\bar{x}^1 \sin\bar{x}^2) + (\sin\bar{x}^2)(\bar{x}^1 \cos\bar{x}^2)$$

$$+ \quad 0 \qquad = 0 = g_{21}$$

Similarly we can prove that

$$g_{13} = g_{31} = 0 \qquad g_{23} = g_{32} = 0$$

$$g_{11} = \frac{\partial x^1}{\partial \bar{x}^1} \frac{\partial x^1}{\partial \bar{x}^1} + \frac{\partial x^2}{\partial \bar{x}^1} \frac{\partial x^2}{\partial \bar{x}^1} + \frac{\partial x^3}{\partial \bar{x}^1} \frac{\partial x^3}{\partial \bar{x}^1}$$

$$= (\cos\bar{x}^2)(\cos\bar{x}^2) + (\sin\bar{x}^2)(\sin\bar{x}^2) + 0$$

$$= 1$$

$g_{22} = r^2 \qquad g_{33} = 1$

Spherical Polar Co-ordinates

In spherical polar Co-ordinates (r, θ, ϕ) we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

If $x^1 = r \quad x^2 = \theta \quad x^3 = \phi \longrightarrow (4)$

$$ds^2 = (dx^1)^2 + r^2 (dx^2)^2 + r^2 \sin^2\theta \, d\phi^2 \longrightarrow (5)$$

Comparing (3) & (5)

$g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta$

$g_{12} = 0 = g_{21} \qquad g_{23} = g_{32} = 0 \qquad g_{31} = g_{13} = 0$

Therefore in matrix form metric tensor can be written as

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

In general for orthogonal Co-ordinates

$$g_{ij} = 0 \quad \text{for} \quad i \neq j$$

By Muhammad Hassain Lecturer (Maths). Govt. College Asghar Mall RWP.

Question #(i) Express the determinant

$$g = |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$

in terms of the elements in 2nd row and their corresponding co-factors

(ii) Show that $g_{ij} G^{ij} = g$ or $g_{ij} G(i,j) = g$ where $G(i,j) = G^{ij}$ is co factor element g_{ij} in g and the summation is only on j.

Sol #

Co-factor of $g_{21} = (-1)^{2+1} \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix}$

Co-factor of $g_{22} = (-1)^{2+2} \begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix}$

Co-factor of $g_{23} = (-1)^{5} \begin{vmatrix} g_{11} & g_{12} \\ g_{31} & g_{32} \end{vmatrix}$

Denoting these Co-factors by G^{21}, G^{22}, G^{23} or $G(2,1), G(2,2), G(2,3)$.

Then by principle of determinant

$$g_{21} G^{21} + g_{22} G^{22} + g_{23} G^{23} = g$$

$$\Rightarrow g_{2i} G^{2i} = g$$

(ii) By applying the result (i) to any row or Column, we have

$g_{ij} G^{ij} = g$ where the summation

is over j only. Thus results holds where $g = |g_{ij}|$ is an nth order det

Question # (i) Prove that

$$g_{21} G^{31} + g_{22} G^{32} + g_{23} G^{33} = 0$$

(ii) prove that

$$g_{ij} G^{kj} = 0 \quad \text{if } i \neq k.$$

Sol Consider the determinant

$$\begin{vmatrix} g_{11} & g_{12} & g_{12} \\ g_{21} & g_{22} & g_{23} \\ g_{21} & g_{22} & g_{23} \end{vmatrix} = 0$$

because 2nd and 3rd rows are identical

Expanding according to the elements of the last row, we have

$$g_{21} G^{31} + g_{22} G^{32} + g_{23} G^{33} = 0$$

(ii) By setting the corresponding elements of any two rows (or columns) equal we can show as in (i)

$$g_{ij} G^{kj} = 0 \quad \text{if } i \neq k$$

Note that this result holds for nth order determinant as well

By Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall.

Conjugate or Reciprocal Tensors

Let $g = |g_{ij}|$ denote the determinant with elements g_{ij} and suppose that $g \neq 0$. Define g^{ij} by

$$g^{ij} = \frac{\text{Co-factor of } g_{ij} \text{ in } |g_{ij}|}{|g_{ij}|}$$

$$= \frac{\text{Co factor of } g_{ij} \text{ in } g}{g}.$$

$$g^{ij} = \frac{G^{ij}}{g} \longrightarrow (1)$$

Then g^{ij} is a symmetric Covariant tensor of Rank 2 called the conjugate or reciprocal tensor of g_{ij}

Note The corresponding Reciprocal Relation equation (1) is

$$g_{ij} = \frac{\text{Co-factor of } g^{ij} \text{ in } |g^{ij}|}{|g^{ij}|}$$

Theorem # If $g^{ij} = \frac{G^{ij}}{g}$ where G^{ij} is the co-factor of g_{ij} in the determinant $g = |g_{ij}| \neq 0$, prove that $g_{ij} g^{kj} = \delta_i^k$

Proof# We know that

$$g_{ij} G^{ij} = g \quad \text{where}$$

summation is over j only

or $$\frac{g_{ij} G^{ij}}{g} = 1$$

or $$g_{ij} \frac{G^{ij}}{g} = 1$$

$$g_{ij} g^{ij} = 1 \longrightarrow (1)$$

Also $$g_{ij} G^{kj} = 0 \quad \text{if } k \neq i$$

$$\Rightarrow \frac{g_{ij} G^{kj}}{g} = 0$$

$$g_{ij} g^{kj} = 0 \quad \text{if } k \neq i \longrightarrow (2)$$

By (1) and (2) we get

$$g_{ij} g^{kj} = 1 \quad \text{if } k = i$$

$$= 0 \quad \text{if } k \neq j$$

$$= \{ \delta_i^k$$

Thus $$g_{ij} g^{kj} = \delta_i^k$$

Theorem# prove that g^{ij} is a symmetric Contravariant tensor of rank two

Proof# If A^p is an arbitrary Contravariant vector, then by inner

product $g_{pj} A^p = A_j$ (say) is an arbitrary covariant vector.

$$g_{pj} A^p = A_j$$

Multiplying by g^{ij}, we get

$$g^{ij} A_j = g^{ij} g_{pj} A^p$$

$$= \delta^i_p A^p$$

$$g^{ij} A_j = A^i$$

Since A_j is an arbitrary covariant vector, g^{ij} is a contravariant tensor of order two, by the application of the Quotient laws. The tensor g^{ij} is called conjugate metric tensor.

Note # Define $g_{ij} g^{kj} = \delta^k_i$

Then δ^k_i is a mixed tensor of order two, contravariant in k and co-variant in i

Theorem # If the Co-ordinate system is orthogonal i.e $g_{ij} = 0$ $i \neq j$, then prove that $g^{ii} = \frac{1}{g_{ii}}$ (no summation over i)

Proof # Since $g_{ij} = 0$ for $i \neq j$, we have

$$g = |g_{ij}| = \begin{vmatrix} g_{11} & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_{nn} \end{vmatrix}$$

$$= g_{11} g_{22} - - - - g_{nn}$$

Now g^{ii} = Co-factor of g_{ii} in g / g

$$= \frac{g_{11} g_{22} - - - g_{i-1} g_{i-1} g_{i+1} g_{i+1} - - g_{nn}}{g_{11} g_{22} - - - - g_{i-1\,i-1} g_{ii} g_{i+1\,i+1} - g_{nn}}$$

$$= \frac{1}{g_{ii}}$$

For an orthogonal system, we get

$$g^{11} = \frac{1}{g_{11}} \qquad g^{22} = \frac{1}{g_{22}} \qquad g^{33} = \frac{1}{g_{33}}$$

Alternate Method#

We know that $g_{ij} g^{kj} = \delta_i^k$

If $i = k = 1$, we get

$$g_{1j} g^{1j} = g_{11} g^{11} + g_{12} g^{12} + g_{13} g^{13}$$

$$\delta_1^1 = g_{11} g^{11} + 0 + 0$$

$$1 = g_{11} g^{11}$$

$$\Rightarrow g^{11} = \frac{1}{g_{11}}$$

Similarly if $i = k = 2$, we get

$$g^{22} = \frac{1}{g_{22}}$$

if $i = k = 3$ $\qquad g^{33} = \frac{1}{g_{33}}$

Conjugate Metric Tensor in Cylindrical and Spherical Polar Co-ordinates

(a) Cylinderical Polar Co-ordinates#

In cylinderical co-ordinates, we have

$$g = |g_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = r^2$$

$$g^{11} = \frac{G^{11}}{g} = \frac{1}{r^2}\begin{vmatrix} r^2 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$g^{22} = \frac{G^{22}}{g} = \frac{1}{r^2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{r^2}$$

$$g^{33} = \frac{G^{33}}{g} = \frac{1}{r^2}\begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix} = 1$$

$$g^{12} = \frac{G^{12}}{g} = \frac{1}{r^2}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

Similarly $g^{ij} = 0$ for $i \neq j$

In matrix form, the conjugate metric tensor can be represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By Muhammad Hussain. Lecturer (Maths) Govt. College Asghar Mall.

Spherical Polar Co-ordinates

In spherical polar Co-ordinates

$$g = |g_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix}$$

$$= r^4\sin^2\theta$$

$$g^{11} = \frac{G^{11}}{g} = \frac{1}{r^4\sin^2\theta}\begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} = 1$$

$$g^{22} = \frac{G^{22}}{g} = \frac{1}{r^4\sin^2\theta}\begin{vmatrix} 1 & 0 \\ 1 & r^2\sin^2\theta \end{vmatrix} = \frac{1}{r^2}$$

$$g^{33} = \frac{G^{33}}{g} = \frac{1}{r^4\sin^2\theta}\begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix}$$

$$= \frac{1}{r^2\sin^2\theta}$$

$g^{ij} = 0$ for $i \neq j$

and in matrix form this can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}$$

By Muhammad Hussain Lecturer (Maths). Govt. College Asghar Mall RWP

Question # Find (i) g and (ii) g^{ij} Corresponding to the Riemannian metric

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6\,dx^1 dx^2 + 4\,dx^2 dx^3$$

Solution Comparing it with metric

$$ds^2 = g_{ij}\,dx^i dx^j \qquad i, j = 1, 2, 3$$

$$= g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{12}\,dx^1 dx^2 + g_{13}\,dx^1 dx^3 + g_{21}\,dx^2 dx^1 + g_{23}\,dx^2 dx^3 + g_{31}\,dx^3 dx^1 + g_{32}\,dx^3 dx^2$$

we have

$$g_{11} = 5 \qquad g_{22} = 3 \qquad g_{33} = 4$$

$$2g_{12} = -6 \Rightarrow g_{12} = -3 = g_{21}$$

$$2g_{13} = 0 \Rightarrow g_{13} = 0 = g_{31}$$

$$2g_{23} = 0 \Rightarrow g_{23} = 0 = g_{32}$$

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 4$$

(ii) The Co-factors G^{ij} of g_{ij} are

$$G^{11} = 8 \qquad G^{22} = 20 \qquad G^{33} = 6$$

$$G^{12} = G^{21} = 12 \qquad G^{23} = G^{32} = 10 \qquad G^{13} = G^{31} = -6$$

$g^{11} = \frac{G^{11}}{g} = \frac{8}{4} = 2$

$g^{22} = \frac{G^{22}}{g} = \frac{20}{4} = 5$

$g^{33} = \frac{G^{33}}{g} = \frac{6}{4} = \frac{3}{2}$

$g^{12} = g^{21} = 3 \qquad g^{23} = g^{32} = -\frac{5}{2}$

$g^{13} = g^{31} = -\frac{3}{2}$

We note that the product of the matrices (g_{ij}) & (g^{ij}) is a unit matrix

Associated Tensors

A tensor obtained from a given tensor by the process inner multiplication of given tensor with g_{ij} or g^{ij} is called a tensor associated with a given tensor.

Effect of inner Multiplication of Metric tensor with other Tensors.

(1) Let A^j be a Contravariant vector. Then the inner product $g_{ij} A^j$ is a covariant tensor denoted by A_i.

$A_i = g_{ij} A^j$ lowering of the suffix of a vector

Thus the process of inner product with a metric tensor has lowered a Contravariant index to a Covariant index

(2) Let A_j be co-variant vector. Then. $g^{ij} A_j$ is a Contravariant vector. we denote it by A^i and is called. associated tensor of A_j

Thus.

$$A^i = g^{ij} A_j$$

Raising of Covariant index to a Contravariant index.

(3) Now Consider an arbitrary 2nd order Contravariant tensor A^{mn}, we have

$$A^n_i = g_{mi} A^{mn}$$
$$A^m_i = g_{in} A^{mn}$$
$$A_{ij} = g_{mi} g_{nj} A^{mn}$$

These three tensors A^n_i, A^m_i and A_{ij} are called associated tensors of A^{mn}

(4) Similarly Consider an arbitrary 2nd order Covariant tensor A_{mn}, we can form the following tensors by raising the indices.

$$A^i_n = g^{mi} A_{mn} \qquad A^i_m = g^{in} A_{mn}$$
$$A^{ij} = g^{mi} g^{nj} A_{mn}$$

(5) The process of raising and lowering indices can be applied to tensors of any order. e.g. Consider a tensor A_{mnp}

$$A^{i}_{np} = g^{mi} A_{mnp}$$

$$A^{i}_{mp} = g^{ni} A_{mnp}$$

$$A^{i}_{mn} = g^{pi} A_{mnp}$$

All these tensors are associated

(6) The process of inner multiplication of g_{ij} or g^{ij} with any tensor say A^{pqr}_{st} lowers or raises the index w.r.t which summation is performed.

Thus $$A^{pq}_{sti} = g_{ri} A^{pqr}_{st}$$

and. $$A^{pqi}_{s} = g^{ti} A^{pqr}_{st}$$

The process of raising and lowering indices is clearly reversible

Note In rectangular Co-ordinates

$$g_{ij} = 1 \quad \text{if } i = j$$

$$= 0 \quad i \neq j$$

So that $A_i = A^i$

This explains why no distinction is made between Contravariant and Covariant

Components of a vector in case of Cartesian tensors

Theorem# If A_i and A^j are the associated tensors of rank 1 and $A_i = g_{ij} A^j$, prove that

$$A^j = g^{ij} A_i$$

Proof# Given that

$$A_i = g_{ij} A^j$$

Multiplying both sides by g^{ik}, we get

$$g^{ik} A_i = g^{ik} g_{ij} A^j$$

$$= g^k_j A^j = A^k$$

$$\Rightarrow A^k = g^{ik} A_i$$

or $$A^j = g^{ij} A_i$$

Theorem# if A^i and A_j are associated tensors of rank 1 and $A^i = g^{ij} A_j$, prove that $A_j = g_{ij} A^i$

Proof# Given that $A^i = g^{ij} A_j$

Multiplying both sides by g_{ik} we get

$$g_{ik} A^i = g_{ik} g^{ij} A_j = \delta^j_k A_j = A_k$$

Christoffel Symbols

Christoffel symbols are certain combinations of partial derivatives of of fundamental tensor g_{ij}. The christoffel symbols are not tensor in general but are very useful in the development of the calculus of tensors.

Christoffel Symbol of the 1st Kind

The christoffel symbols of 1st kind are denoted by $[ij, k]$ and is defined by

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$i, j, k = 1, 2, \cdots n$$

These are called 3-index symbols of 1st kind

Theorem # Prove that

(i) $[ij, k] = [ji, k]$ (ii) $\frac{\partial g_{ij}}{\partial x^k} = [ik, j] + [jk, i]$

Proof #

L.H.S $= [ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$

$= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k}\right) = [ji, k]$

(ii) $[ik, j] = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right)$ $\rightarrow$ (1)

and

$[jk\ \ i] = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$ $\rightarrow$ (2)

Adding (1) & (2)

$$[ik, j] + [jk, i] = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} + \frac{1}{2}\frac{\partial g_{ji}}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k}$$

$$= 2 \cdot H \cdot S$$

Theorem# Show that if $g_{ij} = 0$ for $i \neq j$, then the Christoffel symbols of the 1st kind are given by

(i) $[ii, i] = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^i}$

(ii) $[ii, k] = -\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}$

(iii) $[ij, i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}$

(iv) $[ij, k] = 0$ if i, j, k are distinct

Proof# We know that

$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$ $\rightarrow$ (1)

If $i = j = k$ in (1), we get

$$[ii, i] = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right)$$

$$= \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}$$

(ii) If $i = j \neq h$ in ①

$$[ii, h] = \frac{1}{2}\left(\frac{\partial g_{ih}}{\partial x^i} + \frac{\partial g_{ih}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^h}\right)$$

$$= \frac{1}{2}\left(0 + 0 - \frac{\partial g_{ii}}{\partial x^h}\right)$$

$$= -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^h}$$

(iii) if $i = h \neq j$ in ①, we get

$$[ij, i] = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^i}\right)$$

$$= \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}$$

(iv) If $i \neq j \neq h$ (i.e if all i, j, h are distinct), then from ①

$$[ij, h] = 0$$

Christoffel Symbol of the 2nd Kind

The Christoffel symbol of 2nd kind is denoted by $\left\{ {m \atop ij} \right\}$ and are defined as

$$\left\{ {m \atop ij} \right\} = g^{mh}[ij, h] \quad (i, j, h = 1, 2, \cdots n)$$

The other symbol used instead of $\left\{ {m \atop ij} \right\}$ is Γ^m_{ij}

Theorem# Prove that

(i) $\left\{ {m \atop ij} \right\} = \left\{ {m \atop ji} \right\}$

(ii) $[ij,k] = g_{km}\left\{ {m \atop ij} \right\}$

(iii) $\frac{\partial g^{ij}}{\partial x^k} = -g^{im}\left\{ {j \atop km} \right\} - g^{jm}\left\{ {i \atop km} \right\}$

Proof# (i) we have by definition.

$$\left\{ {m \atop ij} \right\} = g^{mk}[ij,k]$$
$$= g^{mk}[ji,k] = \left\{ {m \atop ji} \right\}$$

(ii) $g_{nm}\left\{ {m \atop ij} \right\} = g_{nm}\, g^{mk}[ij,k]$

$$= \delta_n^k [ij,k]$$

$$= [ij,n]$$

$\Rightarrow [ij,n] = g_{nm}\left\{ {m \atop ij} \right\}$

i.e $[ij,k] = g_{km}\left\{ {m \atop ij} \right\}$ (replacing n by k)

(iii) We know that

$$g_{im}\, g^{mj} = \delta_i^j$$

Diff w.r.t x^k, we get

$$\frac{\partial}{\partial x^k}\left(g_{im}\, g^{mj}\right) = \frac{\partial}{\partial x^k}\left(\delta_i^j\right)$$

$$\Rightarrow g_{im}\frac{\partial}{\partial x^k}\left(g^{mj}\right) + g^{mj}\frac{\partial}{\partial x^k}(g_{im}) = 0$$

$$\Rightarrow g_{im}\frac{\partial g^{mj}}{\partial x^k} = -g^{mj}\frac{\partial g_{im}}{\partial x^k}$$

$$= -g^{mj}\{[ik,m]+[mk,i]\}$$

Multiplying both sides by g^{in}, we get

$$g^{in}g_{im}\frac{\partial g^{mj}}{\partial x^k} = -g^{in}g^{mj}([ik,m]+[mk,i])$$

$$\delta^n_m\frac{\partial g^{mj}}{\partial x^k} = -g^{in}g^{mj}[ik,m] - g^{in}g^{mj}[mk,i]$$

$$\frac{\partial g^{nj}}{\partial x^k} = -g^{in}\begin{Bmatrix} j \\ ik \end{Bmatrix} - g^{mj}\begin{Bmatrix} n \\ mk \end{Bmatrix}$$

Replacing the dummy index in the 1st term on R.H.S, we get

$$\frac{\partial g^{nj}}{\partial x^k} = -g^{mn}\begin{Bmatrix} j \\ mk \end{Bmatrix} - g^{mj}\begin{Bmatrix} n \\ mk \end{Bmatrix}$$

Replacing n by i

$$\frac{\partial g^{ij}}{\partial x^k} = -g^{mi}\begin{Bmatrix} j \\ mk \end{Bmatrix} - g^{mj}\begin{Bmatrix} i \\ mk \end{Bmatrix}$$

$$= -g^{im}\begin{Bmatrix} j \\ mk \end{Bmatrix} - g^{jm}\begin{Bmatrix} i \\ km \end{Bmatrix}$$

By Muhammad Hussain
Lecturer (Maths) Govt. College
Asghar Mall RWP

Theorem# Show that if $g_{ij}=0$ for $i\neq j$, then the Christoffel symbols of 2nd kind are given by

(i) $\left\{ {i \atop ii} \right\} = \frac{1}{2}\frac{\partial}{\partial x^i}\log g_{ii}$

(ii) $\left\{ {i \atop jj} \right\} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i}$

(iii) $\left\{ {i \atop ij} \right\} = \frac{1}{2}\frac{\partial}{\partial x^j}\log g_{ii}$

(iv) $\left\{ {m \atop ij} \right\} = 0$ if i, j and m are distinct, where we suspend the summation convention.

Proof# we know that

$$\left\{ {m \atop ij} \right\} = g^{mk}[ij,k] \longrightarrow (1)$$

Also we have $g^{ii} = \frac{1}{g_{ii}}$ (not summed) $\longrightarrow (2)$

(i) if $i=j=m$ in (1)

$$\left\{ {i \atop ii} \right\} = g^{ik}[ii,k] = g^{ii}[ii,i]$$

$$= \frac{1}{g_{ii}}\left(\frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x^i}(\log g_{ii})$$

(ii) If $i=j\neq m$ in (1), we get

$$\left\{ {m \atop ii} \right\} = g^{mk}[ii,k] = g^{mk}\left(-\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k}\right)$$

Replacing the index m by i

$$\left\{ \begin{matrix} i \\ jj \end{matrix} \right\} = -\frac{1}{2} g^{ik} \frac{\partial g_{jj}}{\partial x^k} = -\frac{1}{2} g^{ii} \frac{\partial g_{jj}}{\partial x^i}$$

$$= -\frac{1}{2 g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$$

(iii) if $i = m \neq j$ in (i) ②, we get

$$\left\{ \begin{matrix} i \\ ij \end{matrix} \right\} = g^{ik} [ij, k] = g^{ii} [ij, i]$$

$$= g^{ii} \left(\frac{1}{2} \frac{\partial g_{ii}}{\partial x^j} \right)$$

$$= \frac{1}{2} \left(\frac{1}{g_{ii}} \frac{\partial g_{ii}}{\partial x^j} \right)$$

$$= \frac{1}{2} \frac{\partial}{\partial x^j} \log g_{ii}$$

(iv) If $i \neq j \neq m$ in (i) ③, we get

$$\left\{ \begin{matrix} m \\ ij \end{matrix} \right\} = g^{mk} [ij, k] = 0$$

i.e when i, j, and m are all distinct, then $\left\{ \begin{matrix} m \\ ij \end{matrix} \right\} = 0$

Theorem# prove that $\left\{ \begin{matrix} i \\ ij \end{matrix} \right\} = \frac{\partial}{\partial x^j} (\log \sqrt{g})$

Proof# we know that

$$g = |g_{ij}| = g_{ij} G^{ij}$$ (Summed over → (i) j fixed)

Since G^{ij} does not contain g_{ij}

, explicitly, therefore differentiating equation (1) w.r.t g_{ij}, we get

$$\frac{\partial g}{\partial g_{ij}} = G^{ij}$$

Also since g is a function of g_{ij} and g_{ij} is function of x^k

$$\frac{\partial g}{\partial x^k} = \frac{\partial g}{\partial g_{ij}} \frac{\partial g_{ij}}{\partial x^k}$$

$$= G^{ij} \frac{\partial g_{ij}}{\partial x^k}$$

$$= g g^{ij} \frac{\partial g_{ij}}{\partial x^k} \quad [\because g^{ij} = \frac{G^{ij}}{g}]$$

$$= g g^{ij} \{ [ik, j] + [jk, i] \}$$

$$= g \left(\left\{ \begin{matrix} i \\ ik \end{matrix} \right\} + \left\{ \begin{matrix} j \\ jk \end{matrix} \right\} \right)$$

$$= 2g \left\{ \begin{matrix} i \\ ik \end{matrix} \right\}$$

Thus $\frac{1}{2g} \frac{\partial g}{\partial x^k} = \left\{ \begin{matrix} i \\ ik \end{matrix} \right\}$

or $\left\{ \begin{matrix} i \\ ik \end{matrix} \right\} = \frac{\partial}{\partial x^k} (\log \sqrt{g})$